5

NONLINEAR OSCILLATORS

Introduction. It is—for a reason evident already in FIGURE 1 of Chapter 3 —*only in the small amplitude approximation* that a one-dimensional system trapped in the neighborhood of a point of stable equilibrium can be expected to approximate a "simple harmonic oscillator." Injection of energy into such a system increases the amplitude of its oscillations,[1] causing the particle to begin to explore regions where the force law differs from that of an idealized spring. From

$$U(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{3}u_2x^3 + \tfrac{1}{4}u_3x^4 + \tfrac{1}{5}u_4x^5 + \tfrac{1}{6}u_5x^6 + \cdots$$

we obtain

$$F(x) = -kx - u_2x^2 - u_3x^3 - u_4x^4 - u_5x^5 \cdots$$

and it is to a study of the physical implications of the red terms—terms that acquire significance only at relatively high energy/amplitude—that we now turn. Such terms **introduce nonlinearity into the equations of motion**, so we will be looking into the *theory of* <u>*nonlinear*</u> *oscillators*. We anticipate that the theory, in at least some of its aspects, will prove to be relatively difficult—that on occasion we will have to bring into play some of the methods of **perturbation theory** and to rest content with results that are only approximately accurate—for nonlinearity deprives us of access to the linear mathematics which has previously figured so importantly in our work. And we anticipate that numerical methods will figure more prominently in our work than they have heretofore.

We begin by looking to some of the qualitative basics of the problem before us.

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[1] Recall that for a simple oscillator

$$\text{energy} = \tfrac{1}{2}k\cdot(\text{amplitude})^2$$

For non-simple oscillators the energy/amplitude relation is, as will emerge, more complicated, but it remains true that increased energy $\Longrightarrow$ increased amplitude.

1. Qualitative basics. We expect low-order nonlinearities to assume importance before those of higher order. In the simplest instance we have

$$U(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{3}u_2x^3 \quad \text{giving} \quad F(x) = -kx - u_2x^2$$

which the most striking feature (FIGURE 1) is the asymmetry of the potential, which causes the non-harmonic part of the force to be directed always to the left (if $u_2 > 0$, and always to the right if $u_2 < 0$). The situation is exposed most clearly when one looks (FIGURE 2) to the contours which the equations

$$\text{energy } \tfrac{1}{2m}p^2 + U(x) = \text{constant}$$

inscribe on phase space. In constructing the figure I have set $\frac{1}{2m} = \frac{1}{2}k = 1$ and $\frac{1}{3}u_2 = \frac{1}{4}$.

More interesting in some ways is the quartic case

$$U(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{4}u_3x^4 \quad \text{giving} \quad F(x) = -kx - u_3x^3$$

(Bilaterally symmetric) scattering/escape are again features of the physics if $u_3 < 0$ (see FIGURES 3 & 4), but more commonly encountered are cases with $u_3 > 0$, for which all states are spatially confined/oscillatory (FIGURE 5).

It will be appreciated that the potentials discussed above are in their own ways no less idealized that the harmonic potential $U(x) = \frac{1}{2}kx^2$, for they speak of forces that become ever stronger as one ventures into regions increasingly remote from the origin. Which—unless, perhaps, one is talking about "classical quarks"—is unphysical. In real-world physics one expects at large amplitude to have additional nonlinear corrections come into play, the net effect being that

$$\lim_{x\to\pm\infty} \big[F(x) = -U'(x)\big] = 0$$

Look, for example to the case of a simple pendulum. The potential is

$$\begin{aligned} U(\theta) &= mg\ell\big[1 - \cos\theta\big] &&: \quad \text{exactly} \\ &\approx mg\ell\big[\tfrac{1}{2}\theta^2\big] &&: \quad \text{in leading (harmonic) approximation} \\ &\approx mg\ell\big[\tfrac{1}{2}\theta^2 - \tfrac{1}{24}\theta^4\big] &&: \quad \text{harmonic approximation + quartic correction} \end{aligned}$$

FIGURE 6 shows plainly that the quartic correction is useful only in a quite restricted energy range, and that it grossly misrepresents the large amplitude physics. Note also that in this instance $U'(\theta)$ does *not* become asymptotically flat —not too surprisingly, for θ refers not to linear separation but to an *angle*.

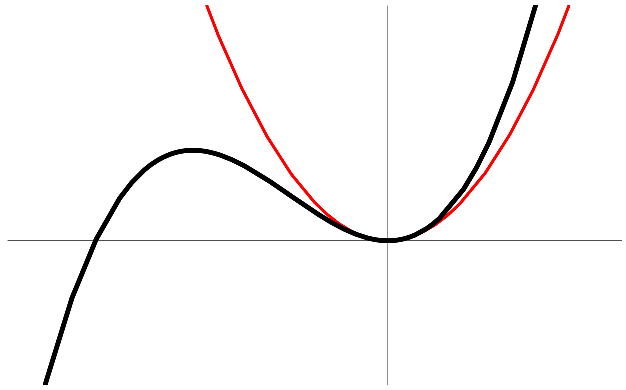

FIGURE 1: *Graph of the typical cubic potential* $U(x) = x^2 + \frac{1}{4}x^3$. *Shown in red, for purposes of comparison, is the harmonic potential* $U(x) = x^2$. *The cubic potential is extremal at* $x = 0$ *and* $x = -\frac{8}{3}$. *At the "top of the hill" it has value* $U(-\frac{8}{3}) = \frac{64}{27}$. *A particle with energy* $E > \frac{64}{27}$ *is not confined to the neighborhood of the origin, but escapes to the left, traveling ever faster.*

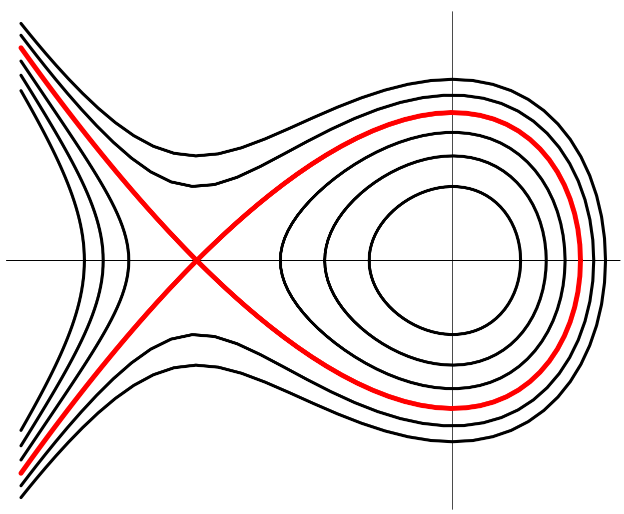

FIGURE 2: *Implicit plot of* $E(p, x) = p^2 + \left[x^2 + \frac{1}{4}x^3\right]$, *inscribed on the phase plane. The red contour arises from setting* $E = \frac{64}{27}$. *Particles with energy* $E > \frac{64}{27}$ *are scattered by the potential: they approach from the left, loop around the origin, exit to the left. Particles with energy* $E < \frac{64}{27}$ *are of two types: those which approach from the left and are scattered back to the left before they get to the origin, and those which are trapped in the neighborhood of the origin.*

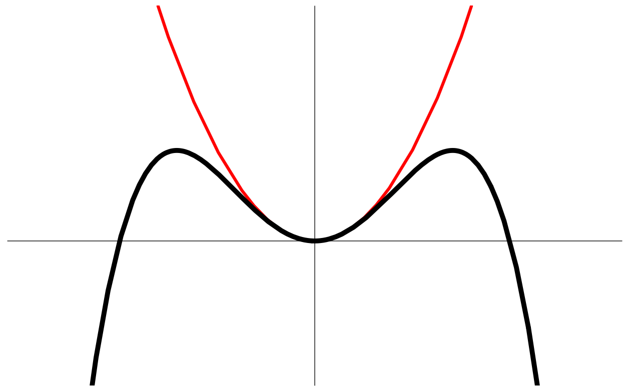

FIGURE 3: *Graph of the typical quartic potential* $U(x) = x^2 - \frac{1}{4}x^4$. *Shown in red, for purposes of comparison, is the harmonic potential* $U(x) = x^2$. *The quartic potential is extremal at* $x = 0$ *and* $x = \pm\sqrt{2}$. *At the "top of the hills" it has value* $U(\pm\sqrt{2}) = 1$. *Only particles with energy* $E < 1$ *are confined to the neighborhood of the origin.*

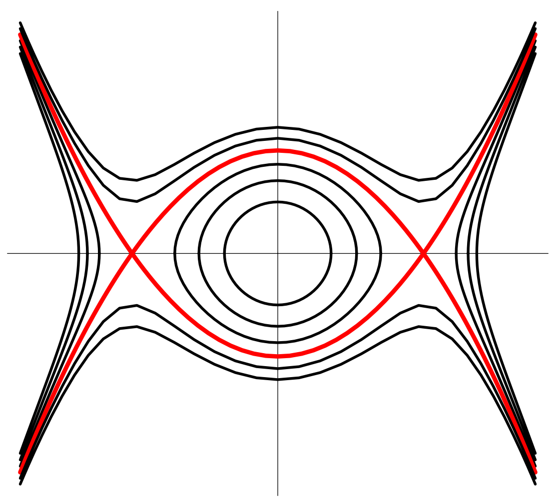

FIGURE 4: *Implicit plot of* $E(p, x) = p^2 + \left[x^2 - \frac{1}{4}x^4\right]$, *inscribed on the phase plane. The red contour arises from setting* $E = 1$. *Particles with energy* $E > 1$ *approach from the left (right), do a little jig near the origin, then continue to the right (left). Particles with energy* $E < 1$ *are of two types: those which approach from the left (right) and are scattered back to the left (right) before they get to the origin, and those which are trapped in the neighborhood of the origin.*

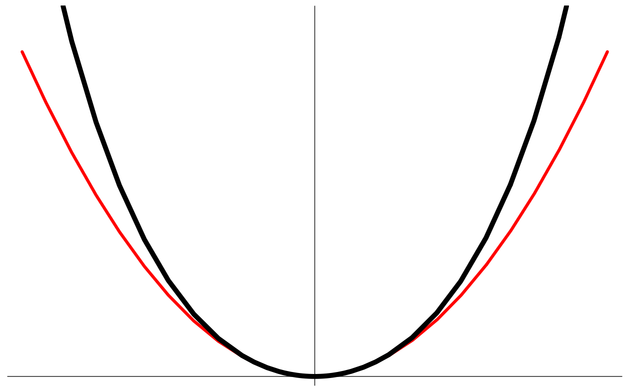

FIGURE 5: *Graph of the typical quartic potential* $U(x) = x^2 + \frac{1}{20}x^4$. *Shown in red, for purposes of comparison, is the harmonic potential* $U(x) = x^2$. *In this case scattering does not occur: all orbits, however great the energy of the particle, are symmetrically bounded on left and right.*

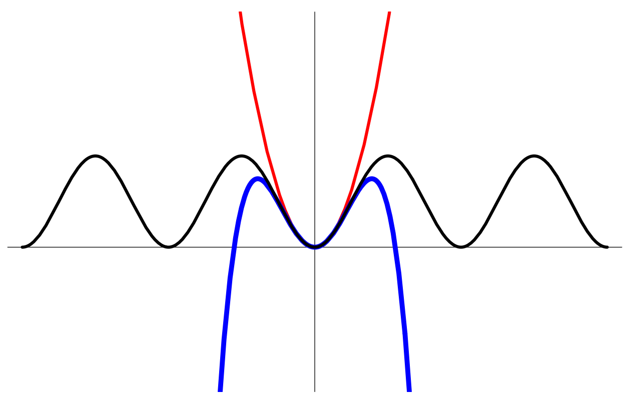

FIGURE 6: *Shown in black is the exact pendulum potential* $U(\theta)$, *in red the small amplitude harmonic approximation to that potential, and in blue the harmonic potential with quartic correction. It is evident (see again page 2) that inclusion of the quartic correction is a useful refinement only if the amplitude/energy are not too large.*

2. Nonlinearity implies anharmonicity. A mass point m moves within a potential well, as shown above. From energy conservation $\frac{1}{2}m\dot{x}^2 + U(x) = E$ it follows that the time dt required for the particle to move from x to $x + dx$ can be described

$$dt = \frac{dx}{\sqrt{\frac{2}{m}\big[E - U(x)\big]}}$$

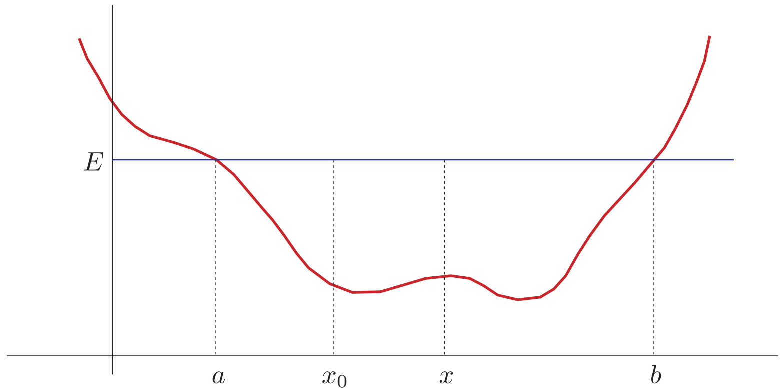

FIGURE 7: *A mass point m moves with conserved energy E in the presence of the potential indicated by the red curve. Its motion is necessarily confined to the interval bounded by the* **turning points** *a and b, for outside of that interval one would have*

$$\text{kinetic energy} = \tfrac{1}{2}m(\text{speed})^2 < 0$$

which would force the speed to be imaginary!

The **time of flight** (or "transit time")

$$x_0 \xrightarrow[\text{energy } E]{} x$$

is given therefore by

$$\mathcal{T}_E(x_0 \longmapsto x) = \int_{x_0}^{x} \frac{1}{\sqrt{\frac{2}{m}\big[E - U(y)\big]}}\,dy$$

Such one-dimensional motion is necessarily periodic, with

$$\begin{aligned}\text{period } \tau_E &= 2\mathcal{T}_E(a \longmapsto b)\\ &= 2\int_a^b \frac{1}{\sqrt{\frac{2}{m}\big[E - U(y)\big]}}\,dy\end{aligned} \tag{1}$$

EXAMPLE: Harmonic potential. From the symmetry of the harmonic potential $U(x) = \frac{1}{2}kx^2$ it follows that the turning points are symmetrically placed, at (let us say) $\pm A$. The energy is $E = \frac{1}{2}kA^2$, so we have

$$\begin{aligned}\tau_E &= 2\int_{-A}^{+A} \frac{1}{\sqrt{\frac{k}{m}\big[A^2 - y^2\big]}}\,dy\\ &= \frac{2}{\omega}\int_{-1}^{+1} \frac{1}{\sqrt{1 - z^2}}\,dz \quad \text{with} \quad z \equiv y/A,\ \ \omega \equiv \sqrt{k/m}\\ &= \frac{2}{\omega}\,\text{Arcsin}(x)\Big|_{-1}^{+1}\\ &= 2\pi/\omega \quad \underline{\text{for all values of } A}\end{aligned}$$

This **energy-independence of the period** is a property special to the harmonic oscillator, and is in fact the *reason* that such oscillators are said to be "harmonic."

Look to the **dimensional analysis** of the situation. From $U(x) = \frac{1}{2}kx^n$ we get $[k] = M^1L^{2-n}T^{-2}$ so

$$\tau = m^p k^q A^r \quad \text{entails} \quad \begin{cases} p+q=0 \\ q(2-n)+r=0 \\ -2q=1 \end{cases}$$

giving $q = -\frac{1}{2}$, $p = +\frac{1}{2}$, $r = 1-\frac{1}{2}n$: τ will be amplitude independent only in the case $n = \frac{1}{2}$.

EXAMPLE: General evenpower-law potential. Here the potential is taken to have the form $U(x) = \frac{1}{2}kx^n$ with n even (odd powers to not produce potential wells, so play no role in the present discussion). Arguing as before, we obtain

$$\tau_{A,n} = \frac{4}{\sqrt{k/m}A^{\frac{n}{2}-1}}\int_0^1 \frac{1}{\sqrt{1-z^n}}\,dz$$

The integrals lead, according to *Mathematica*, to ratios of gamma functions:

$$\begin{aligned} \tau_{A,2} &= \frac{1}{\sqrt{k/m}A^0}\,2\pi\cdot \\ \tau_{A,4} &= \frac{1}{\sqrt{k/m}A^1}\,2\pi\cdot 0.834627 \\ \tau_{A,6} &= \frac{1}{\sqrt{k/m}A^2}\,2\pi\cdot 0.773064 \\ &\vdots \\ \tau_{A,n} &= \frac{1}{\sqrt{k/m}A^{\frac{n}{2}-1}}\,4\sqrt{\pi}\,\frac{\Gamma(1+\frac{1}{n})}{\Gamma(\frac{1}{2}+\frac{1}{n})} \end{aligned}$$

Heavy integration supplied the numerical details, but dimensional analysis was by itself powerful enough to supply the rest.

3. Cubic perturbation. We are concerned with one-dimensional systems that in the simplest instance[2] possess equations of motion of the form

$$m\ddot{x} = -kx + \varepsilon(\text{nonlinear terms})$$

In the preceding section we identified the condition that must prevail if the motion $x(t)$ of the oscillator is to be periodic: $x(t) = x(t+\tau)$. And we discovered

[2] Later we will want to include terms that describe damping and harmonic stimulation.

how, in each individual periodic case, to compute the *energy-dependent value* of τ. The more interesting physics lives, however, in the finer details, and to glimpse those one must have in hand some representative *solutions* of the equations of motion. Partial information about those can be obtained analytical perturbation theory, but the computational detail tends to be so dense as to obscure the qualitative essence of the physics.[3] It is to avoid those tedious distractions that I will make heavy use of *Mathematica*'s powerful numerical capabilities.

We will concern ourselves here with systems of the type

$$U(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{3}m\alpha x^3$$

which yield

$$\ddot{x} + \omega_0^2 x + \alpha x^2 = 0 \quad : \quad \omega_0^2 \equiv k/m$$

To make a perturbation-theoretic approach feasible one would insist that $\alpha \ll 1$, but the numerical approach imposes no such restriction.[4] In the case $k = m = 2$, $\alpha = \frac{3}{8}$ we have $U(x) = x^2 + \frac{1}{4}x^3$, the case to which FIGURES 1 & 2 refer. The equation of motion becomes

$$\ddot{x} + x + \tfrac{3}{8}x^2 = 0$$

We will restrict our attention to cases in which $x(0) = 0$. Were we to set

initial kinetic energy = potential energy at top of the hill

we would have $\dot{x}(0) = \sqrt{64/27} = 1.5396$: the motion will be bounded/periodic if $0 < \dot{x}(0) < 1.5396$, but if $\dot{x}(0) > 1.5396$ the particle will escape over the top of the potential hill at its first opportunity. Look now to FIGURE 8, which was constructed

```
harmonic=x[t]/.
First[NDSolve[{x''[t]+x[t]==0, x[0]==0, x'[0]==1.0},
x[t], {t,0,2π}]]

cubic050=x[t]/.
First[NDSolve[{x''[t]+x[t]+3/8 x[t]^2==0, x[0]==0, x'[0]==0.55},
x[t], {t,0,2π}]]
```

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[3] See, for example, Chapter 7 in I. G. Main, *Vibrations and waves in physics* (3^{rd} edition 1993). One circumstance that complicates the analytical study of nonlinear differential equations stems from the elementary fact that if $z = x+iy$ then

$$\text{real part of } (x+iy)^n \neq x^n \text{ unless } n = 1$$

The powerful "complex variable trick" is thus rendered inapplicable. And, of course, we **lose the principle of superposition**, which is a grievous loss.

[4] Note, however, that as α increases the potential well becomes ever shallower, and the energy range that leads to periodic motion becomes ever narrower.

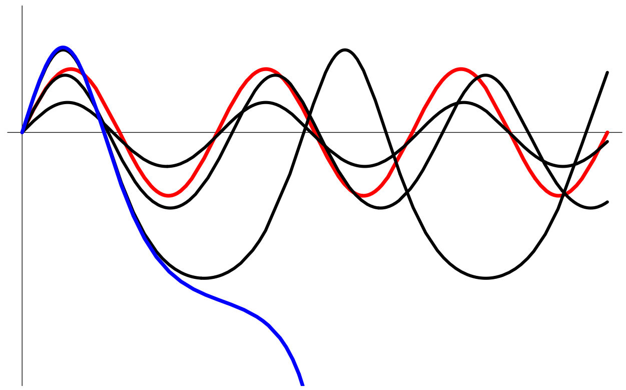

FIGURE 8: *Motion of a mass $m = 2$ in the presence of the cubically perturbed potential $U(x) = x + \frac{1}{4}x^3$ shown in* FIGURE 1. *The initial velocities were taken to be $\dot{x}(0) = 0.5, 1.0, 1.5$ (all less that the critical launch velocity $\dot{x}_{\text{critical})}(0) = 1.5396$) and $\dot{x}(0) = 1.55$. In the latter (blue) instance the particle escapes. Notice that the positive excursions become shorter as they become more energetic—as one might anticipate on the basis of the discussion in §2. But that trend is contradicted by the negative excursions, for the reason that particles of greater energy approach nearer to the top of the hill, where they move more slowly. The red curve shows typical motion in the absence of the cubic term.*

`cubic100`, `cubic150` and `cubic155` are constructed similarly. Finally we command

```
Plot[Evaluate[{harmonic, cubid050m cubic100,cubic150,cubic155}],
{t,0,6π}, PlotRange → {−4,2}, Ticks → False,
PlotStyle → {{RGBColor[1,0,0],Thickness[0.006]},
              {RGBColor[0,0,0],Thickness[0.005]},
              {RGBColor[0,0,0],Thickness[0.005]},
              {RGBColor[0,0,0],Thickness[0.005]},
              {RGBColor[0,0,1],Thickness[0.006]}}];
```

Similar command sequences will be used to construct subsequent figures.

4. Upturned quartic perturbation. We turn now to systems of the type

$$U(x) = \tfrac{1}{2}kx^2 + \tfrac{1}{4}m\alpha x^4 \quad : \quad \alpha \geqslant 0$$

which yield

$$\ddot{x} + \omega_0^2 x + \alpha x^3 = 0$$

From the upturned symmetry $U(x) = U(-x)$ of the potential it follows that the motion of the particle is invariably bounded by symmetrically-placed turning

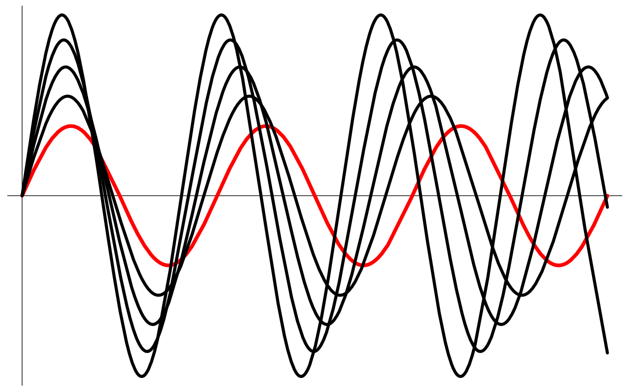

FIGURE 9: *Graphs that result from* (2) *when* $x(0) = 0$ *and the initial velocity is assigned the values* $\dot{x}(0) = 1.5, 2.0, 2.5, 3.0$. *The red curve is typical of those that result when the quartic perturbation is turned off. Note that increasing the energy decreases the period, consistently with our experience on page* 7.

points $\pm A$, periodic $x(t + \tau_A) = x(t)$, and that $x(t + \frac{1}{2}\tau_A) = -x(t)$. If we set $m = k = 2$ and $\alpha = \frac{1}{10}$ the potential becomes that depicted in FIGURE 5 and the equation of motion becomes

$$\ddot{x} + x + \tfrac{1}{10}x^3 = 0 \tag{2}$$

Numerically-generated solutions are shown in the preceding figure.

PROBLEM 1: Construct the modified figure that would result from introducing a linear damping term into (2):

$$\ddot{x} + 2\gamma\dot{x} + x + \tfrac{1}{10}x^3 = 0$$

Set $2\gamma = \frac{1}{8}$. And notice that, if we are going to abandon linearity anyway, it might be reasonable to look into the consequences of the **nonlinear damping** that would be achieved by inclusion of terms of the form $\dot{x}^{\text{odd power}}$. The physically important and much-studied **Rayleigh-van der Pohl equation**

$$\ddot{x} + \omega_0^2 = \epsilon(\dot{x} - \tfrac{1}{3}\dot{x}^3)$$

provides and example.

The energy of such an oscillator (assuming that $x(0) = 0$) can be described

$$E = \tfrac{1}{2}m[\dot{x}(0)]^2 = \tfrac{1}{2}kA^2[1 + \tfrac{1}{2}\tilde{\alpha}A^2] \quad : \quad \tilde{\alpha} \equiv \alpha/\omega_0^2 = m\alpha/k$$

Returning with this information to (1) we obtain

$$\tau_A = 4\int_0^A \frac{1}{\sqrt{\frac{2}{m}\left[\frac{1}{2}kA^2[1+\frac{1}{2}\tilde{\alpha}A^2] - \frac{1}{2}ky^2[1+\frac{1}{2}\tilde{\alpha}y^2]\right]}}\,dy$$
$$= \frac{4}{\omega_0}\int_0^1 \frac{1}{\sqrt{[1+\beta]-z^2[1+\beta z^2]}}\,dz \quad : \quad \beta \equiv \tfrac{1}{2}\tilde{\alpha}A^2$$

Using `Series[`*integrand*`,{`β`,0,2}]` to expand the integrand, then integrating term-wise, we find

$$\tau_A = \frac{4}{\omega_0}\Big\{\frac{\pi}{2} - \frac{3\pi}{8}\beta + \frac{57\pi}{128}\beta^2 + \cdots\Big\}$$
$$= \tau_0\Big\{1 - \tilde{\alpha}\frac{3A^2}{8} + \tilde{\alpha}^2\frac{57A^4}{256} + \cdots\Big\} \quad : \quad \tau_0 \equiv 2\pi/\omega_0$$

which—it is reassuring to observe—does assume the correct value at $\alpha = 0$, and does show the correct diminishing trend when α is small. By algebraic inversion

$$\begin{aligned}\omega_A &\equiv 2\pi/\tau_A \\ &= \omega_0\big\{1 + \alpha\tfrac{3}{8}(A/\omega_0)^2 - \alpha^2\tfrac{21}{256}(A/\omega_0)^4 + \cdots\big\} \\ &= \omega_0 + \omega_1 + \omega_2 + \cdots\end{aligned} \tag{3}$$

It might appear on casual inspection that FIGURE 9 refers to functions of the form $x(t) = A\sin\omega_A t$. But such functions clearly do not satisfy (2). We are led therefore to contemplate solutions of the form[5]

$$x(t) = a_1\sin\omega_A t + a_3\sin 3\omega_A t + a_5\sin 5\omega_A t + \cdots \tag{4}$$

and to have $A = a_1 - a_3 + a_5 - a_7 + \cdots$. To lend detailed substance to those anticipatory remarks one turns to perturbation theory.

Perturbation theories—whatever the context in which they are encountered (celestial mechanics, quantum mechanics, . . .)—entail chains of calculation that can invariably be organized in a variety of distinct ways, each with its own advantages/disadvantages. A. H. Nayfeh & D. T. Mook, in their splendid monograph,[6] treat no fewer than four distinct variants of the perturbation

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[5] Even terms are excluded on the ground that their presence would violate the requirement that the resulting curves be *symmetrical with respect to their extrema*: compare the figures that result from

```
Plot[{Sin[t], Sin[3t, Sin[t]+1/10 Sin[3t]}, {t,0,2π}];
```

and

```
Plot[{Sin[t], Sin[2t, Sin[t]+1/10 Sin[2t]}, {t,0,2π}];
```

[6] *Nonlinear Oscillations* (1979), §2.3, pages 50–61.

theory of nonlinear one-dimensional oscillators. The following discussion is based upon a method developed by A. Lindstedt & H. Poincaré.

4. Perturbation theory of an oscillator with quartic nonlinearity. To gain leverage on the problem we in place of $\ddot{x} + \omega_0^2 x + \alpha x^3 = 0$ write

$$\ddot{x} + \omega_0^2 x + \epsilon\alpha x^3 = 0 \tag{5}$$

which smoothly interpolates between the problem of interest ($\epsilon = 1$) and its linear companion ($\epsilon = 0$). We have interest in periodic functions of

$$s \equiv \omega t$$

so will write

$$x(t) = z(s) \quad \text{which entails} \quad \ddot{x}(t) = \omega^2 z''(s)$$

And—with the figure in mind—we declare ourselves to be interested only in functions that conform to the initial condition $x(0) = z(0) = 0$.

The equation of motion now reads

$$\omega^2 \cdot z'' + \omega_0^2 z + \epsilon\alpha z^3 = 0$$

Into this we introduce

$$\begin{aligned} \omega &= \omega_0 + \epsilon\,\omega_1 + \epsilon^2\omega_2 + \cdots \\ z &= z_0 + \epsilon\, z_1 + \epsilon^2 z_2 + \cdots \end{aligned}$$

expand and set the terms of order ϵ^0, of order ϵ^1, of order $\epsilon^2, \ldots$ separately equal to zero. This gives

$$\omega_0^2\big[z_0'' + z_0\big] = 0 \tag{6.0}$$
$$\omega_0^2\big[z_1'' + z_1\big] = -2\omega_0\omega_1 z_0'' - \alpha z_0^3 \tag{6.1}$$
$$\omega_0^2\big[z_2'' + z_2\big] = -2\omega_0\omega_1 z_1'' - (\omega_1^2 + 2\omega_0\omega_2) z_0'' - 3\alpha z_0^2 z_1 \tag{6.2}$$
$$\vdots$$

which we undertake to solve serially. From (6.0) and our declared initial condition (which will be enforced at each individual step of the procedure) we have

$$z_0(s) = \mathcal{A}\sin(s) \tag{7.0}$$

Proceeding with that information to (6.1) we have

$$\omega_0^2\big[z_1'' + z_1\big] = 2\omega_0\omega_1\mathcal{A}\sin s - \alpha\mathcal{A}^3\sin^3 s$$

which when solved (use `DSolve`) subject to the condition $z(0) = 0$ gives

$$z_1(s) = C_1 \sin s + (\text{six terms of the form } \cos ps \sin qs) + \underbrace{s \cdot \frac{\big[12\alpha\mathcal{A}^2 - 32\omega_0\omega_1\big]}{32\omega_0^2} a_0 \cos s}_{\text{“secular term”}}$$

Here C_1 is a constant of integration which we set equal to zero on the grounds that it would otherwise bring into play a term redundant with $z_0(s)$. The so-called **secular term**, if allowed to remain, **would violate periodicity**: to kill it we set

$$\omega_1 = \alpha\omega_0 \tfrac{3}{8}(\mathcal{A}/\omega_0)^2 \tag{8.1}$$

Next we use (8.1) to eliminate all reference to ω_1 and, proceeding one term at a time, we `TrigReduce` each of the $\cos ps \sin qs$ terms (this is accomplished by highlighting such a term and then hitting the `TrigReduce` button on the `AlgebraicManipulation` palette). Grouping similar terms, we obtain finally

$$z_1(s) = -\alpha\mathcal{A}(\mathcal{A}/\omega_0)^2\big[\tfrac{1}{32}\sin s + \tfrac{1}{32}\sin 3s\big] \tag{7.1}$$

which, as we readily verify, is in fact a particular solution of (6.1).

Next we introduce (7.1), (8.1) and (7.2) into (6.2) and proceed exactly as before: we set the new constant of integration equal to zero (for the same reason as before) and to kill the

$$\text{new secular term} = -s \cdot \frac{2\mathcal{A}[21\alpha^2\mathcal{A}^4 + 256\omega_0^3\omega_2]}{512\omega_0^4}\cos s$$

we set

$$\omega_2 = -\alpha^2\omega_0\tfrac{21}{512}(\mathcal{A}/\omega_0)^4 \tag{8.2}$$

Thus are we led finally (after the familiar `TrigReduce` procedure) to

$$z_2(s) = \alpha^2\mathcal{A}(\mathcal{A}/\omega_0)^4\big[\tfrac{21}{1024}\sin s + \tfrac{3}{128}\sin 3s + \tfrac{1}{1024}\sin 5s\big] \tag{7.2}$$

which, as we readily verify, is in fact a particular solution of (6.2).

We now set ϵ—which has done its work—equal to unity[7] and have

$$\begin{aligned} x(t) = \mathcal{A}\Big\{ &\sin(\omega t) - \alpha(\mathcal{A}/\omega_0)^2\big[\tfrac{1}{32}\sin\omega t + \tfrac{1}{32}\sin 3\omega t\big] \\ &+ \alpha^2(\mathcal{A}/\omega_0)^4\big[\tfrac{21}{1024}\sin\omega t + \tfrac{3}{128}\sin 3\omega t + \tfrac{1}{1024}\sin 5\omega t\big] + \cdots\Big\} \end{aligned} \tag{9.1}$$

with

$$\omega = \omega_0\big[1 + \alpha\tfrac{3}{8}(\mathcal{A}/\omega_0)^2 - \alpha^2\tfrac{21}{512}(\mathcal{A}/\omega_0)^4 + \cdots\big] \tag{9.2}$$

The latter equation is, is will be noticed, in precise agreement with (3), which was derived by other means. Or would be if we could identify $\mathcal{A}$ with the

[7] Our results remain valid/useful when looked upon as expansions in powers of the small parameter α. If the quartic adjustment of our harmonic potential were in fact not "weak," if it were too large to be treated as a "perturbation," then we would have to adopt an altogether different (numerical?) mode of analysis.

amplitude A. Equation (9.1) describes, however a periodic function with

$$\begin{aligned}\text{amplitude } A &= x\big(\tfrac{\pi}{2}\big)\\ &= \mathcal{A}\big[1 - \alpha^2\tfrac{1}{512}(\mathcal{A}/\omega_0)^4 + \cdots\big]\end{aligned}$$

Returning with this information—note the absence of a term of order α^1—to (3) we recover (9.2), which is to say: (3) and (9.2) are in precise agreement *through terms of second order in* α, which is all the agreement we can ask for, since that is the order in which we have been working.

How well have we done? If we assign to m, k, ω_0 and α the values ($m = k = 2$, $\omega_0 = 1$, $\alpha = \frac{1}{10}$) that were used to construct FIGURE 9 then (9.1) becomes

$$\begin{aligned}x(t) = A\Big\{ &\sin(t) - \tfrac{1}{10}A^2\big[\tfrac{1}{32}\sin t + \tfrac{1}{32}\sin 3t\big]\\ &+ \tfrac{1}{100}A^4\big[\tfrac{21}{1024}\sin t + \tfrac{3}{128}\sin 3t + \tfrac{1}{1024}\sin 5t\big] + \cdots\Big\}\end{aligned}$$

And if we assign to A the values 1.42884, 1.84835, 2.23607 and 2.59484 that by $\dot{x}(0) = \sqrt{\frac{2}{m}\big[\frac{1}{2}kA^2 + \frac{1}{4}m\alpha A^4\big]}$ correspond to $\dot{x}(0) = 1.5, 2.0, 2.5$ and 3.0, and if finally we superimpose graphs of the resulting functions $x(t)$ upon a duplicate of FIGURE 9, we obtain FIGURE 10, the seeming implication being that we have done very well indeed!

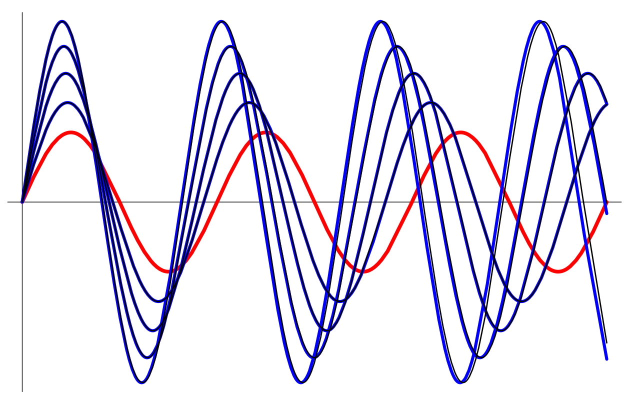

FIGURE 10: *Superimposed here upon a red and blue copy of* FIGURE 9 *—which was produced by numerical analysis—are graphs of the corresponding of the instances of the* $x(t)$ *of* (9.1)*, which was obtained by* 2nd *order perturbation theory.*

Perturbation theory—here as always, whatever the field of application—is invariably computationally challenging, if carried to higher than leading order. But the computations are of a sort that can readily delegated to *Mathematica*, and that could be accomplished effortlessly by special purpose program if one were motivated (which one seldom is) to carry the work to 3rd, 4th or higher order. That is a lesson of general significance, but the insight gained from our effort is mainly qualitative:

- Nonlinearity tends generally to cause period/frequency to become energy/amplitude-dependent (but this is a lesson learned already in §2);
- Aperiodic "secular terms" were found to arise at every order, and it was our effort to kill those terms (to restore periodicity) that supplied the information we used to construct a description of *how* frequency depends upon amplitude (but it was remarked already in §2 that the time-of-flight formula (1) provides a much more general and direct approach to the solution of that problem);
- Perturbation theory has supported the conjecture introduced at (4). We found more specifically that

  a $\sin 3\omega t$-term (+ an additional $\sin\omega t$-term) was introduced in 1st order

  We anticipate that

  a $\sin 5\omega t$-term (+ additional $\sin\omega t$ and $\sin 3\omega t$-terms) will be introduced in 2st order

  and expect that pattern to continue.
- We recognize that those odd harmonics—which are reminiscent of the harmonics of an organ pipe that is open on one end—originated in these simple trigonometric identities

$$\begin{aligned}
\texttt{TrigReduce}[\cos 2s\cdot\sin s] &= \tfrac{1}{2}\big\{-\sin s+\sin 3s\big\}\\
\texttt{TrigReduce}[\cos s\cdot\sin 2s] &= \tfrac{1}{2}\big\{+\sin s+\sin 3s\big\}\\
\texttt{TrigReduce}[\cos s\cdot\sin 4s] &= \tfrac{1}{2}\big\{+\sin 3s+\sin 5s\big\}\\
\texttt{TrigReduce}[\cos^2 2s\cdot\sin s] &= \tfrac{1}{4}\big\{+2\sin s-\sin 3s+\sin 5s\big\}\\
&\vdots
\end{aligned}$$

  and that the products on the left stem from the nonlinearity of the equation of motion.

5. Resonances of a forced nonlinear oscillator. We have been looking to the perturbation-theoretic solution of the equation of motion

$$\ddot{x}+\omega_0^2 x+\epsilon\alpha x^3=0 \tag{5}$$

of what—somewhat confusingly—I have called "oscillators with weak quartic nonlinearity." Confusingly because, while a quartic does appear in the potential

$$U(x)=\tfrac{1}{2}m\omega_0^2x^2+\epsilon\tfrac{1}{4}m\alpha x^4$$

it contributes a cubic to the equation of motion. The time has come, I think, to assign to (5) the name by which it is commonly known: (5) was first discussed by G. Duffing (1918), and has been intensively studied—partly because it arises from the lowest-order nonlinear term in the expansion of the functions $U(x)$ that refer to *symmetric potential wells*, partly because it serves so well to illustrate the properties of nonlinear oscillators *in general*, partly because some of its solutions have been discovered to illustrate phenomena basic to the modern *theory of chaos*. It is called the **Duffing equation**.

The occurance of harmonics in the solutions of (5) suggests that the oscillator might be especially responsive not only to stimuli of frequency $\nu \sim \omega_0$ but also to stimuli of frequencies $\nu \sim 3\omega_0, 5\omega_0, \ldots$ I turn now to description of an argument that lends analytical support to that conjecture.

> **PROBLEM 2**: a) Plot the numerical solution of (10) that arises in the case $\omega_0 = 1$, $\nu = 1.2$, $2\gamma = \frac{1}{10}$, $\alpha = \frac{1}{30}$, $x(0) = \dot{x}(0) = 0$ as t ranges from 0 to 200. Set `PlotRange`$\rightarrow\{-6, 6\}$ and `MaxBend`$\rightarrow 1$. What do you conclude? Construct—for your own edification—similar graphs for assorted values of ν, 2γ and α.
>
> b) Do the same for $150 < t < 200$ and call that graph `response`. Plot $\sin(1.2t)$ for $150 < t < 200$ (with `PlotRange` set as before) and call that graph `stimulus`. `Show[{stimulus, response}]`. What do you conclude?

Experiments such as those just performed establish to our satisfaction that—after transcients have died down, and all initial data has been forgotten—harmonically stimulated nonlinear oscillators (just like linear oscillators) *move not at their natural frequencies, but in phase-shifted synchrony with the stimulus*. It is upon this proposition that we will build.

Having concerned ourselves previously with the homogeneous equation (5), we look now to the inhomogeneous equation of motion

$$\begin{aligned}
\ddot{x} + \omega^2 x + \epsilon(\alpha x^3 + 2\gamma\dot{x}) &= S\sin(\nu t + \delta) \\
&= S\cos\delta \cdot \sin\nu t + S\sin\delta \cdot \cos\nu t \\
&\equiv S_1 \sin\nu t + S_2 \cos\nu t \qquad (10)
\end{aligned}$$

Here ϵ is a bookkeeping device intended to emphasize that we consider both the nonlinearity and the damping to be small, and to enable us to distinguish 1^{st}-order from 2^{nd}-order from 3^{rd}-order ... effects: we will, in point of fact, be working only in 1^{st}-order, and at the end of the day will set $\epsilon = 1$ on grounds that it is really α and γ that are small. In the present context the $_0$ on ω_0 serves no purpose, so will be dropped. The literature records many attempted solutions[8] of (10), but all proceed

[8] See, for example, A. H. Nayfeh & D. T. Mook, *Nonlinear Oscillations* (1979), §4.1; A. H. Nayfeh, *Introduction to Perturbation Techniques* (1981), Chapter 9; J. J. Stoker, *Nonlinear Vibrations in Mechanical & Electrical Systems* (1950), Chapter 4; C. Hayashi, *Nonlinear Oscillations in Physical Systems* (1964); J. V. José & E. J. Salatan, *Classical Mechanics*(1998), §7.1.2.

$$\text{stimulus} \longrightarrow \text{response}$$

and all, as I read them, either are marred by seemingly unmotivated leaps and arbitrary assumptions or are offputtingly complicated. I have discovered that it is very much easier and less problematic to proceed

$$\text{stimulus} \longleftarrow \text{response}$$

—very much in the spirit of **PROBLEM 9** in Chapter 3. We ask for the conditions under which the response

$$x(t) = A\sin\nu t + \epsilon B\sin 3\nu t + \epsilon^2 C\sin 5\nu t + \cdots \tag{11}$$

can be demonstrated to arise from a stimulus of the form $S(t) = S\sin(\nu t + \delta)$.

Introducing (11) into the expression on the left side of (10) we obtain (in first order)

$$A(\omega^2 - \nu^2)\sin\nu t + \epsilon\big[2A\gamma\nu\cos\nu t + B(\omega^2 - 9\nu^2) + A^3\alpha\sin^3\nu t\big] + \cdots$$

But $\sin^3\nu t = \frac{3}{4}\sin\nu t - \frac{1}{4}\sin 3\nu t$ so the preceding expression becomes

$$\big\{A(\omega^2-\nu^2)+\epsilon\tfrac{3}{4}A^3\alpha\big\}\sin\nu t+\epsilon 2A\gamma\nu\cos\nu t+\epsilon\big\{B(\omega^2-9\nu^2)-\tfrac{1}{4}A^3\alpha\big\}\sin 3\nu t+\cdots$$

We force this to resemble the expression on the right side of (10) by setting

$$B = \frac{A^3\alpha}{4(\omega^2 - 9\nu^2)}$$

$$\begin{aligned} S_1 &= A(\omega^2 - \nu^2) + \epsilon\tfrac{3}{4}A^3\alpha \\ S_2 &= \epsilon 2A\gamma\nu \end{aligned}$$

The net implication (if at this point we set $\epsilon = 1$) is that the response

$$x(t) = A\sin\nu t + \frac{A^3\alpha}{4(\omega^2 - 9\nu^2)}\sin 3\nu t + \cdots \tag{12}$$

arises in first order from the stimulus

$$S(t) = S\sin(\nu t + \delta)$$

where

$$S = A\sqrt{\big[(\omega^2 - \nu^2) + \tfrac{3}{4}A^2\alpha\big]^2 + \big[2\gamma\nu\big]^2} \tag{13.1}$$

$$\delta = \arctan\left[\frac{2\gamma\nu}{(\omega^2 - \nu^2) + \frac{3}{4}A^2\alpha}\right] \tag{13.2}$$

From (13.1) we obtain

$$A = \frac{S}{\sqrt{\big[(\omega^2 - \nu^2) + \frac{3}{4}A^2\alpha\big]^2 + \big[2\gamma\nu\big]^2}} \tag{14}$$

Squaring and multiplying by the denominator, we have

$$S^2 - \big[(\omega^2 - \nu^2)^2 + 4\gamma^2\nu^2\big]A^2 - \tfrac{3}{2}\alpha(\omega^2 - \nu^2)A^4 - \tfrac{9}{4}\alpha^2 A^6 = 0$$

which—if we consider the stimulus amplitude S to be given/fixed, and $\{\omega, \gamma, \alpha\}$ to describe given/fixed properties of the damped nonlinear oscillator—presents $A(\nu)$ as the root of cubic polynomial in A^2. Such a polynomial—since the coefficients are real—necessarily has either

- three real (but not necessarily distinct) roots, or
- one real root and two complex roots (that are conjugates of one another).

We infer that $A(\nu)$ may—at some frequencies ν and for some parameter settings —be **triple valued**. The point is illustrated in FIGURE 11.

The multivaluedness of $A(\nu)$ accounts (see FIGURE 12) for an instance of the **jump discontinuities** that are a commonly encountered symptom of nonlinearity. If the stimulus frequency ν is dithered up and down through an interval that includes both ν_{low} and ν_{high} then one can expect to see $A(\nu)$ trace a **hysteresis loop**. T. W. Arnold & W. Case have described[9] a simple mechanical apparatus that serves to illustrate these and other characteristic consequences of nonlinearity. More commonly encountered—in both literature and laboratory—are electrical circuits that demonstrate the effects of nonlinearity.

The preceding discussion refers to the effect of nonlinearity upon the **primary resonance** $\nu \sim \omega$ of a Duffing oscillator. Similar remarks are shown in sources already cited[8] to pertain to the **superharmonic resonances** $\nu \sim 3\omega, 5\omega, 7\omega, \ldots$

It has become conventional to call $A(\nu)$ the "amplitude" of the response function $x(t)$, though it is obvious that to discover the true maximum of

$$x(t) = A(\nu)\sin\nu t + \epsilon B(\nu)\sin 3\nu t + \epsilon^2 C(\nu)\sin 5\nu t + \cdots$$

one would have to take the contribution of the higher-order terms also into account. Far from being negligible, that contribution can be dominant. We learned, for example at (12) that in 1^{st}-order theory

$$B(\nu) = \frac{A^3(\nu)\,\alpha}{4(\omega^2 - 9\nu^2)}$$

which blows up at $\nu = \frac{1}{3}\omega$. A more refined analysis would establish the existence and develop properties of the **subharmonic resonances** $\nu \sim \frac{1}{3}\omega, \frac{1}{5}\omega, \frac{1}{7}\omega, \ldots$ For an accessible account of the details, see pages 104–112 in Stoker.[8]

[9] "Nonlinear effects in a simple mechanical system, " AJP **50**, 220 (1982).

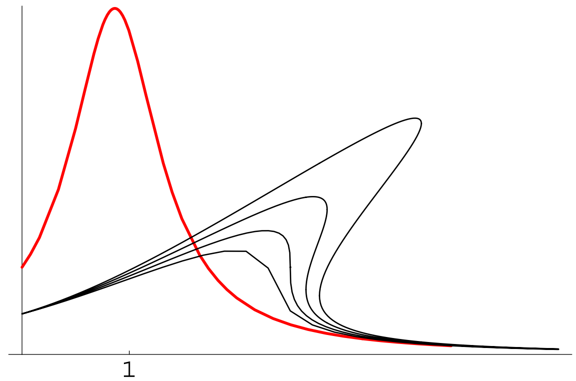

FIGURE 11: *Illustration of the fact that harmonic stimulation of a Duffing oscillator leads to a response-amplitude function $A(\nu)$ that is sometimes triple valued. The figure displays A^2 vs. ν^2, and was obtained from from* (14) *by means of Mathematica's* `ImplicitPlot` *resource. Parameters have been assigned the values $S=\omega=\frac{3}{4}\alpha=1$, ant the slanted peaks have become progressively taller as the damping term $4\gamma^2$ descends through the values $0.4, 0.3, 0.2, 0.1$. Shown in red for purposes of comparison is the amplitude function that results at $4\gamma^2=0.1$ when the nonlinearity has been turned off: $\alpha=0$.*

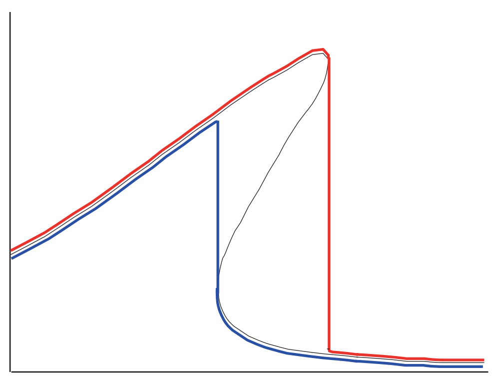

FIGURE 12: *If the stimulation frequency ν is slowly increased, with other parameters held constant, one comes to a point ν_{high} at which $A(\nu)$ abruptly jumps (red curve) to a smaller value. If, on the other hand, ν is slowly decreased one comes to a different/lower point ν_{low} at which $A(\nu)$ abruptly jumps (blue curve) to a higher value.*

6. Combination resonances for two-frequency stimulation. Qualitatively new aspects of nonlinear oscillator physics come into evidence when the stimulus contains more thana one frequency component. To illustrate some of hte points at issue we look to the system (compare (10))

$$\ddot{x} + \omega^2 x + \epsilon(\alpha x^3 + 2\gamma\dot{x}) = S_1 \sin(\nu_1 t) + S_2 \sin(\nu_2 t + \delta)$$

We insert $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 \cdots$ into the preceding equation, expand in powers of ϵ, group together terms of the same order and obtain

$$\ddot{x}_0 + \omega^2 x_0 = S_1 \sin(\nu_1 t) + S_2 \sin(\nu_2 t + \delta) \tag{11.0}$$
$$\ddot{x}_1 + \omega^2 x_1 = -2\gamma\dot{x}_0 - \alpha x_0^3 \tag{11.1}$$
$$\ddot{x}_2 + \omega^2 x_2 = -2\gamma\dot{x}_1 - 3\alpha x_0^2 x_1 \tag{11.2}$$

The general solution of (11.0) is reported by *Mathematica* to be

$$x_0(t) = A\sin(\omega t + \beta) + \frac{S_1}{\omega^2 - \nu_1^2}\sin\nu_1 t + \frac{S_2}{\omega^2 - \nu_2^2}\sin(\nu_2 t + \delta) \tag{12.0}$$

where A and β are arbitrary constants. From this it follows that

$$\texttt{TrigReduce[}\,2\gamma\dot{x}_0 + \alpha x_0^3\,\texttt{]//Simplify} = \text{sum of 31 trigonometric terms}$$

The solution of (11.1) is therefore challenging, but can with patience be done term by term. The result (after another `TrigReduce[]//Simplify`) is found to be of the form

$$x_1(t) = \text{sum of 37 terms}$$

But four of those terms (of which $4A\gamma t\sin[\omega t + \beta]$ is typical) are aperiodic secular terms: to kill them we must set $A = 0$, which serves to kill most of the periodic terms as well. We are left with

$$\begin{aligned}
x_1(t) = B\sin[\,\omega t + \beta\,] &- \frac{2S_1\gamma\nu_1}{(\omega^2 - \nu_1^2)^2}\cos[\nu_1 t] - \frac{2S_2\gamma\nu_1}{(\omega^2 - \nu_2^2)^2}\cos[\nu_2 t + \delta\,] \\
&+ \frac{3S_1^3\alpha}{4(\omega^2 - \nu_1^2)^4}\sin[\nu_1 t] + \frac{3S_2^3\alpha}{4(\omega^2 - \nu_2^2)^4}\sin[\nu_2 t + \delta\,] \\
&+ \frac{3S_1S_2^2\alpha}{2(\omega^2 - \nu_1^2)^2(\omega^2 - \nu_2^2)^2}\sin[\nu_1 t] + \frac{3S_1^2S_2\alpha}{2(\omega^2 - \nu_1^2)^2(\omega^2 - \nu_2^2)^2}\sin[\nu_2 t + \delta\,] \\
&- \frac{S_1^3\alpha}{4(\omega^2 - 9\nu_1^2)(\omega^2 - \nu_1^2)^3}\sin[3\nu_1 t] - \frac{S_2^3\alpha}{4(\omega^2 - 9\nu_2^2)(\omega^2 - \nu_2^2)^3}\sin[3(\nu_2 t + \delta)] \\
&- \frac{3S_1^2S_2\alpha}{4(\omega^2 - \nu_1^2)^2(\omega^2 - \nu_2^2)(\omega^2 - [2\nu_1 + \nu_2]^2)}\sin[(2\nu_1 + \nu_2)t + \delta\,] \\
&- \frac{3S_1S_2^2\alpha}{4(\omega^2 - \nu_1^2)(\omega^2 - \nu_2^2)^2(\omega^2 - [\nu_1 + 2\nu_2]^2)}\sin[(\nu_1 + 2\nu_2)t + 2\delta\,] \\
&- \frac{3S_1^2S_2\alpha}{4(\omega^2 - \nu_1^2)^2(\omega^2 - \nu_2^2)(\omega^2 - [2\nu_1 - \nu_2]^2)}\sin[(2\nu_1 - \nu_2)t - \delta\,] \\
&- \frac{3S_1S_2^2\alpha}{4(\omega^2 - \nu_1^2)(\omega^2 - \nu_2^2)^2(\omega^2 - [\nu_1 - 2\nu_2]^2)}\sin[(\nu_1 - 2\nu_2)t - 2\delta\,]
\end{aligned} \tag{12.1}$$

where it is now B and β that are arbitrary. Using (12) to construct $x = x_0 + \epsilon x_1$, we observe that the system is

- resonant if either $\nu_1 \sim \omega$ or $\nu_2 \sim \omega$;
- resonant if either $\nu_1 \sim \frac{1}{3}\omega$ or $\nu_2 \sim \frac{1}{3}\omega$;
- resonant if either $|2\nu_1 + \nu_2| \sim \omega$ or $|2\nu_1 - \nu_2| \sim \omega$ or $|\nu_1 + 2\nu_2| \sim \omega$ or $|\nu_1 - 2\nu_2| \sim \omega$.

Resonances of the latter sort are called **combination resonances**.

In acoustics, combination resonances—especially those of the $|\nu_1 \pm \nu_2|$ variety—are called **combination tones** (or "third tones"). Hermann Helmholtz (1821–1894), who devotes Chapter 7 of his monumental *On the Sensations of Tone* (1$^{\text{st}}$ edition 1862, 4$^{\text{th}}$ edition 1877) to the subject, states that the phenomenon and its fundamental importance to the perception of musical harmony was first recognized (1714) by Giuseppe Tartini, the Italian violinist and composer (1692–1770), and later stressed (1745) by the German organist and theorist Georg Andreas Sorge (1703–1778). From his Appendix 12 it becomes clear that Helmnoltz understood quite clearly that the perception of combination tones originates in the circumstance that in the presence of loud sounds the ear functions like a nonlinear oscillator:[10] he presents there a sketch of the essentials of precisely the argument that led us to equations (12).

At the beginning of his research career Chandrasekhar Raman (1888–1970), working under the influence of Helmholtz' and Rayleigh's then-recent but already highly influential contributions to the theory of sound, cultivated an interest in the vibrational physics of musical instruments. Among the systems that engaged his attention[11] is the one shown in Figure 13. He was fascinated by the complex vibrational patterns (combinational resonances) that arose when the forks were tuned to distinct frequencies. It would be easy to argue that it was this experience that prepared his mind for the discovery—only a few years later—of Raman scattering/Raman spectroscopy.[12]

Figure 13: *C. V. Raman's experimental set-up. Forks tuned to distinct frequencies stimulated the respective ends of a non-linear string. Stroboscopic examination revealed "combination resonances."*

[10] My own ears are apparently more nonlinear than most: when presented with the sound of a tuning for, I hear chords. I have not attempted to determine whether the spurious frequencies conform to the pattern developed in §5.

[11] My source here is G. Venkataraman, *Journey into Light*: *Life & Science of C. V. Raman* (1988). See especially Chapter 4, pages 75–78.

[12] For a good short account of that subject, see http://carbon.cudenver.edu/public/chemistry/classes/chem4538/raman.htm.

It will be appreciated that the phenomenon under discussion *hinges critically on the nonlinearity* of the equation of motion. If one sets $\alpha = 0$ then (12) reduces to the statement that

$$\begin{aligned} x(t) = B\sin(\omega t + \beta) &+ \Big\{\frac{S_1}{\omega^2 - \nu_1^2}\sin\nu_1 t - \epsilon\frac{2S_1\gamma\nu_1}{(\omega^2-\nu_1^2)^2}\cos\nu_1 t\Big\} \\ &+ \Big\{\frac{S_2}{\omega^2 - \nu_2^2}\sin(\nu_2 t + \delta) - \epsilon\frac{2S_2\gamma\nu_1}{(\omega^2-\nu_2^2)^2}\cos(\nu_2 t + \delta)\Big\} \end{aligned} \tag{13}$$

But at $\alpha = 0$ our equation of motion has become

$$\ddot{x} + \epsilon 2\gamma\dot{x} + \omega^2 x = S_1\sin(\nu_1 t) + S_2\sin(\nu_2 t + \delta)$$

for which (see again page 15 in Chapter 3) we possess the *exact* solution

$$\begin{aligned} x(t) = B\sin(\omega t + \beta) &+ \frac{S_1}{\sqrt{(\omega^2-\nu_1^2)^2 + 4\epsilon^2\gamma^2\nu_1^2}}\sin\Big[\nu_1 t - \arctan\frac{\epsilon 2\gamma\nu_1}{\omega^2-\nu_1^2}\Big] \\ &+ \frac{S_2}{\sqrt{(\omega^2-\nu_2^2)^2 + 4\epsilon^2\gamma^2\nu_2^2}}\sin\Big[\nu_2 t + \delta - \arctan\frac{\epsilon 2\gamma\nu_2}{\omega^2-\nu_2^2}\Big] \end{aligned}$$

Expansion in powers of ϵ gives back (in first order) precisely (13). This little argument serves to expose the specific respects in which the argument that led to (12) is defective: it provides no indication of the adjustment

$$\frac{1}{\omega^2 - \nu^2} \longmapsto \frac{1}{\sqrt{(\omega^2-\nu^2)^2 + 4\epsilon^2\gamma^2\nu^2}}$$

that typically serves to temper the singularities at resonance, and it provides only a veiled indication of the phase shift. If carried to higher order (daunting prospect!) the theory, whether or not it remedied those defects, would pretty clearly lead to additional, more complexly-constructed combination frequencies.

The preceding discussion owes some of its characteristic features to the fact that it was a cubic term αx^3 that we introduced into the equation of motion; had we inserted a quadratic nonlinearity αx^2, as is more commonly done,[13] we would have been led to combination frequencies $|\nu_1 \pm \nu_2|$. And there is, of course, no physical reason for the stimulus $S(t)$ not to be a superposition of *three or more* frequencies $\nu_1, \nu_2, \nu_3, \ldots$ No reason, indeed, for it not to be an *arbitrary* function of time, like the signal delivered to a nonlinear speaker.

7. Numerical methods. We have in recent pages been studying phenomena—particularly resonance phenomena—manifested by driven nonlinear oscillators, systems with equations of motion of the general form

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x + f(x) = S(t) \quad : \quad f(x) \text{ nonlinear}$$

[13] See, for example, §10.1 in A. H. Nayfeh.[8]

One thing has become clear: the *analytical* theory of such systems presents difficulties at every turn. It is well to take note, therefore, of the fact that there is another way: one can proceed numerically. With modern software such an approach can be quick, easy and highly informative. Here I will illustrate the point as it relates to systems of the type

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x + \alpha x^3 = S_1 \sin\nu_1 t + S_2 \sin\nu_2 t \tag{14}$$

Note that I have omitted the familiar ϵ-factors, since we will now *not* be drawing upon perturbation theory.

We recognize first of all that it is very easy to graph the solution of (14) in any particular case:

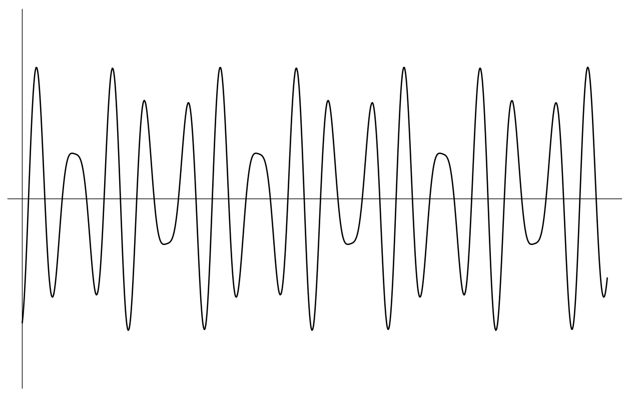

FIGURE 14: *Solution of* (14) *in the case* $S_1 = S_2 = 4$, $\omega = \alpha = 1$, $2\gamma = \frac{1}{10}$, $x(0) = \dot{x}(0) = 0$. *The commands were*

```
motion = NDSolve[{x''[t]+(1/10)x'[t]+x[t]+x[t]^3 == Sin[5t]+Sin[7t],
x[0] == 0, x'[0] == 0}, x[t], {t,80,180}] [[1]]

Plot[x[t] /. motion, {t,160,180}, Ticks→False];
```

We waited until $t = 80$ *to start the evaluation of* $x(t)$ *so that the initial transcients—to which the assumed natural frequency* $\omega = 1$ *contributes prominently—have had a chance to die down. And for clarity we have plotted only the last* 20 *time units.*

We were led by the discussion in §6 to expected resonances at

$$\begin{array}{ll}
\nu_1 = 5 & 2\nu_1 + \nu_2 = 17 \\
\nu_2 = 7 & 2\nu_1 - \nu_2 = 3 \\
3\nu_1 = 15 & 2\nu_2 + \nu_1 = 19 \\
3\nu_2 = 21 & 2\nu_2 - \nu_1 = 9
\end{array}$$

To expose those we look to the **power spectrum** of the computed $x(t)$, which provides indication of the relative weights of the Fourier components that contribute to the construction of $x(t)$.[14] To that end, we command

```
discretizedmotion = Table[x[t] /. motion, {t, 80,180,.05}];
```

and then plot the absolute value of the discrete Fourier transform of the list thus generated:

```
powerspectrum = ListPlot[Take[Abs[Fourier[discretizedmotion]],
{1,200}], PlotJoined → True, PlotRange → {0,0.5},
PlotStyle → Thickness[0.007], Ticks → False];
```

I have removed the ticks because they refer to frequency bin numbers, rather than to literal frequency. To remedy that defect we command

```
referencefreqs = Table[(Sin[t]+Sin[5t]+Sin[7t])/60,{t, 100,200,.05}];
```

and plot (in color) the power spectrum of that data:

```
referencefreqsplot = ListPlot[Take[Abs[Fourier[referencefreqs]],
{1,180}], PlotJoined → True, PlotRange → {0,0.5},
PlotStyle → {Thickness[0.005], RGBColor[1,0,0]}, Ticks → False];
```

Finally we command `Show[{referencefreqsplot,powerspectrum}];` and get the following figure, which shows resonances at the driving frequencies $\nu_1 = 5$, $\nu_2 = 7$ and—just as important—the absence of a transcient resonance at $\omega = 1$.

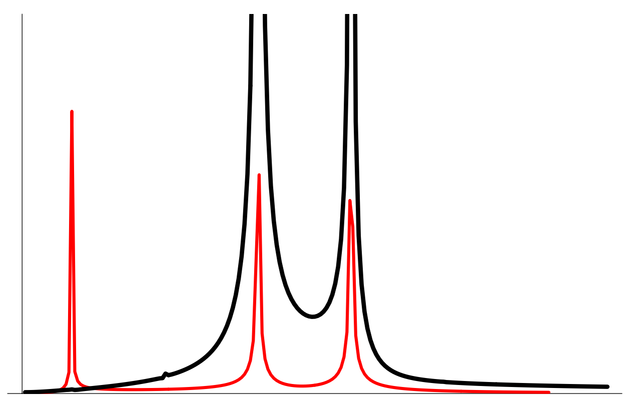

FIGURE 15: *Power spectrum of the motion $x(t)$ shown in* FIGURE 14.

[14] This is a subject to which we will later have occasion to give detailed attention. In the meantime see http://en.wikipedia.org/wiki/Power_spectrum.

The plan now is, while holding $\{S_1, S_2, \alpha, \gamma, \nu_1, \nu_2, x(0), \dot{x}(0)\}$ fixed, to slowly increase ω—to turn FIGURE 15 into a movie, as it were—and to look for the appearance of peaks in the spectral density, peaks announcing that we have come upon a secondary resonance frequency ("secondary resonance" being a term that refers collectively to superharmonics, subharmonics, combination resonances).[15] To illustrate the results to which such a procedure might lead: we are led by 1^{st}-order theory[16] to anticipate a resonance at $2\nu_1 - \nu_2 = 3$, so we set $\omega = 3$ and obtain FIGURES 16 & 17.

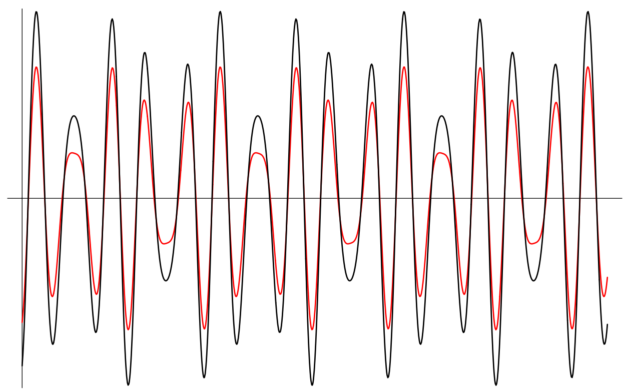

FIGURE 16: *Graph of the solution $x(t)$ of* (14) *when* $\{S_1, S_2, \alpha, \gamma, x(0), \dot{x}(0)\}$ *retain their former values, but* $\omega = 3$. *The graph has been superimposed upon a red copy of* FIGURE 14 *to display the difference between* $x_{\omega=1}(t)$ *and* $x_{\omega=3}(t)$. *The driving frequencies predominate. Variation of the natural frequency* ω*, now that the transcients have died, has served only to change the amplitude.*

The somewhat skewed profile of the resonance at $\omega = 3$ (FIGURE 17) is more pronounced at $\omega = 3.1$ (FIGURE 18), where it has become a distinct "zig-zag." Such a profile is to be expected whenever perturbation theory supplies a factor of the form

$$\frac{1}{(\omega^2 - \omega^2_{\text{resonance}})^{\text{odd}}}$$

—as, indeed, (12.1) does in this instance.[17]

One often hears it said that "nonlinear physics is difficult." The preceding discussion suggests that, while such physics may be difficult to approach *analytically*, it can be expected to yield readily enough to *numerical* analysis.

---

[15] In the laboratory ω—since it refers to an intrinsic property of the oscillator —would typically *not* be susceptible to variation: one would tune ν_1 and/or ν_2.

[16] Of dubious relevance, one might suppose, since at $\alpha = 1$ the nonlinearity is *not* small.

[17] Look to the coefficient of $\sin[(2\nu_1 - \nu_2)t - \delta]$.

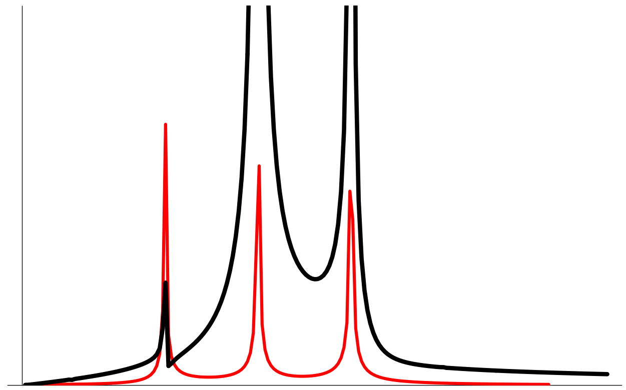

FIGURE 17: *Power spectrum in the case* $\omega = 2\nu_1 - \nu_2 = 3$. *Careful examination of* FIGURE 15 *shows that a faint hint of this resonance was evident already at* $\omega = 1$.

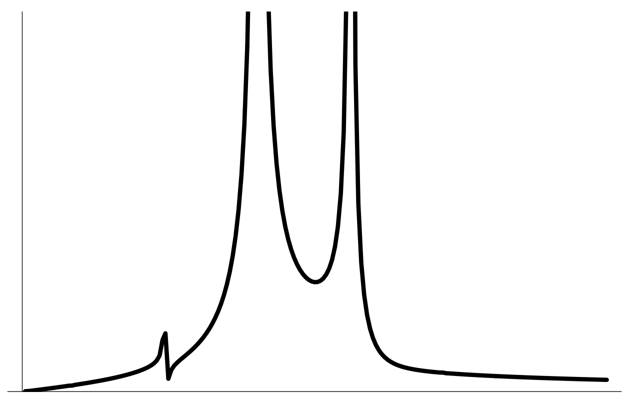

FIGURE 18: *Power spectrum in the case* $\omega = 3.1$.

8. Chaos. Arguing from

$$\text{torque} = \tfrac{d}{dt}(\text{angular momentum})$$

and FIGURE 19 we obtain the pendulum equation

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \quad : \quad \omega^2 \equiv g/\ell$$

In leading nonlinear approximation we have

$$\ddot{\theta} + \omega^2\theta - \tfrac{1}{2}\omega^2\theta^3 = 0$$

which is an instance of the equation we have studied now at some length. I propose now, however, to look (numerically) to the pendulum equation in its

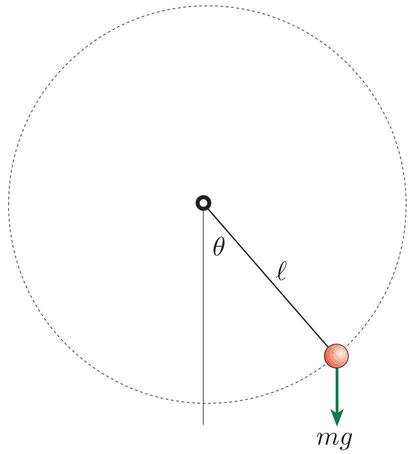

FIGURE 19: *The gravitational torque (relative to the pivot point) is*

$$\text{torque} = -mg\ell\sin\theta$$

The angular momentum of the bob (again relative to the pivot point) is

$$\text{angular momentum} = m\ell^2\dot{\theta}$$

full nonlinear glory. Introducing damping and a harmonic stimulus (by nature a torque, not a force), we have this particular instance

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\sin\theta = S\cos\nu t \tag{15}$$

of the equation presented at the bottom of page 22. What is so striking about (15) is the inexhaustible **variety of its solutions**, of which I must be content to display but a small sample.[18] The graphs of $\theta(t)$ were produced by commands identical to those presented in the caption of FIGURE 14, the only difference being that I allowe t to run from 0 to 250. To obtain parametric plots of the curve traced on the phase plane by $\{\theta(t), \dot{\theta}(t)\}$ the command was

```
ParametricPlot[{Evaluate[x[t] /.  pendulummotion].
Evaluate[D[x[t] /.  pendulummotion,t]]} /.  t→T,
{T,0,250}, MaxBend→1];
```

[18] I have selected my examples from among those discussed by S.Neil Rasband in §6.4 of his *Chaotic Dynamics of Nonlinear Systems* (1990).

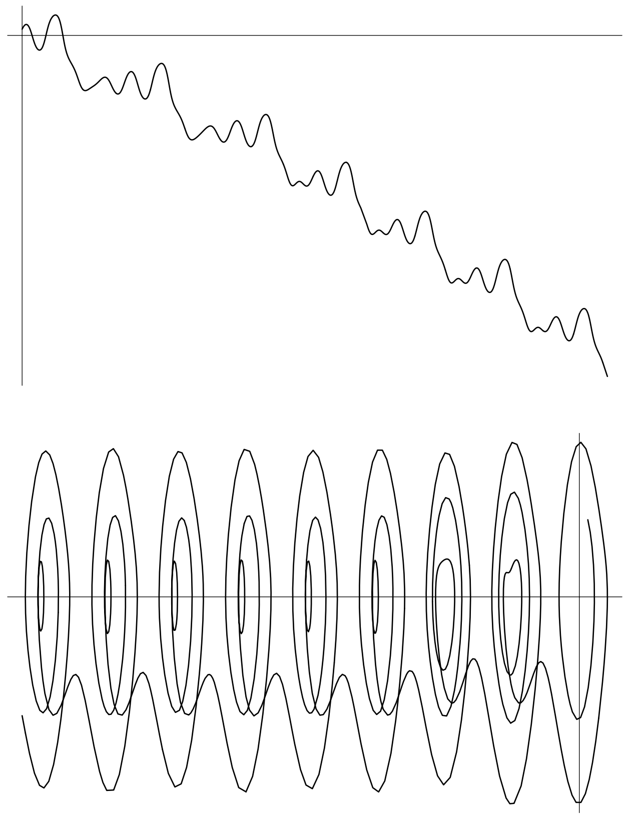

FIGURE 20: *Above*: *a graph of* $\theta(t)$ *in the case*

$$\gamma = \tfrac{1}{10},\ \omega^2 = 1,\ S = 0.52,\ \nu = 0.694,\ \theta(0) = 0.8,\ \dot{\theta}(0) = 0.8$$

Below: *the same data displayed as on the phase plane. The system has discovered a* **period-3 limit cycle**.

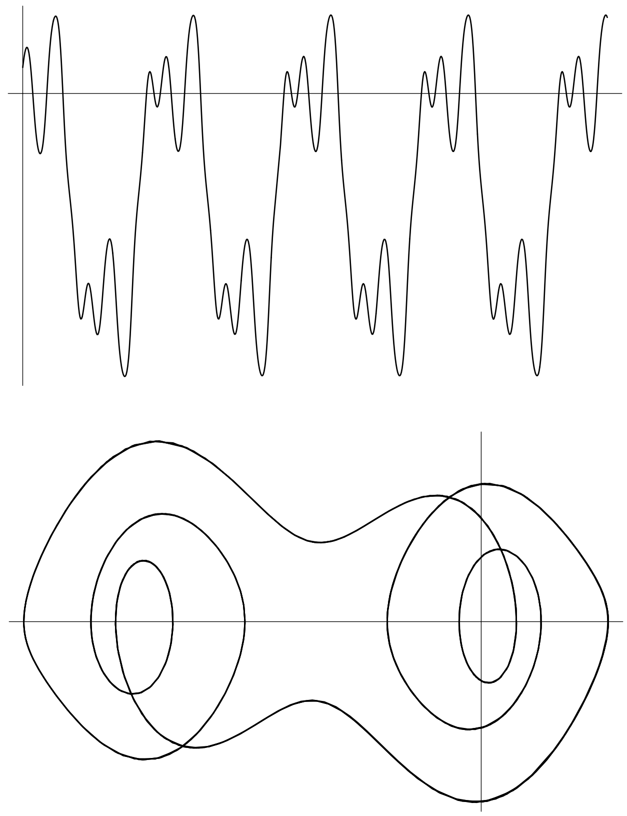

FIGURE 21: *All parameters are the same as in* FIGURE 20 *except that the stimulus frequency has been adjusted* $\nu \longmapsto 0.668$. *The system has discovered a* **period-5 limit cycle**.

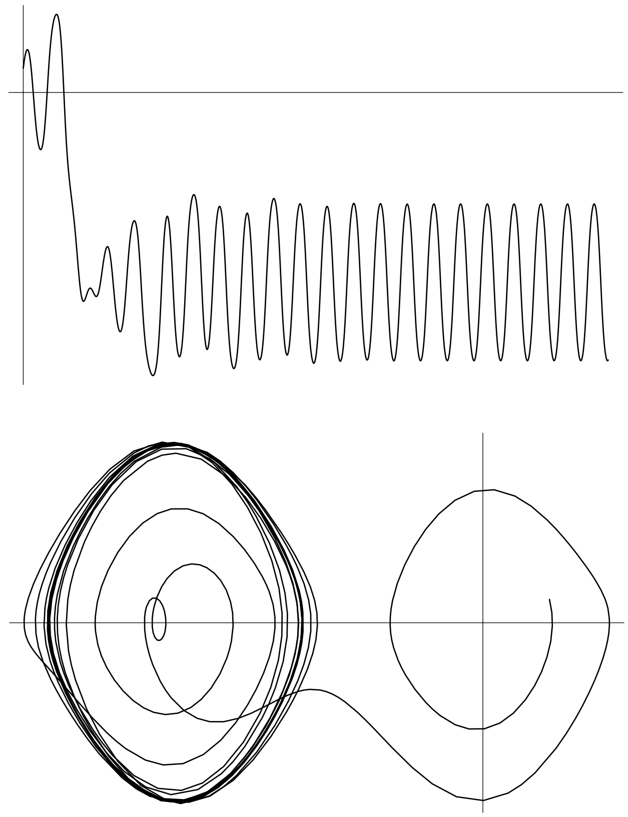

FIGURE 22: *All parameters are the same as in* FIGURE 21 *except that the initial conditions have been adjusted*

$$\theta(0) \longmapsto -0.8, \quad \dot{\theta}(0) \longmapsto 0.1234$$

The system has discovered a **period-1 limit cycle**.

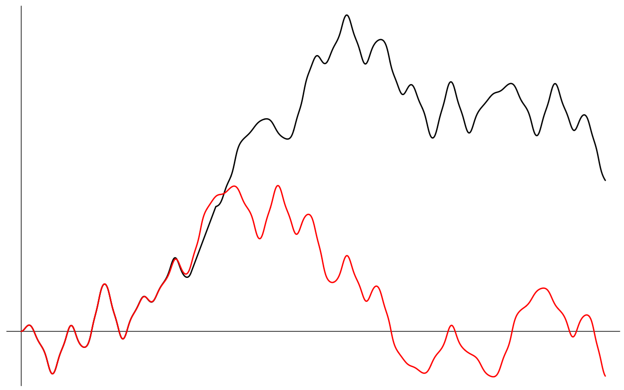

FIGURE 23: *Superimposed graphs of* $\theta(t)$ *in the cases*

$$\gamma = \frac{1}{10},\ \omega^2 = 1,\ S = 0.85,\ \nu = 0.53,\ \theta(0) = \mathbf{0.0000},\ \dot{\theta}(0) = 0$$

and (in red)

$$\gamma = \frac{1}{10},\ \omega^2 = 1,\ S = 0.85,\ \nu = 0.53,\ \theta(0) = \mathbf{0.0002},\ \dot{\theta}(0) = 0$$

Rasband[18] *states that the motion of the pendulum is in this case (and in infinitely many other cases) demonstrably* **chaotic**. *Note the rapid divergence of solutions that proceed from very nearly identical initial conditions.*

I have advanced no technical definition of "chaos," nor do I (on this occasion) intend to. The points I wish to make are simply that

- the harmonically stimulated damped pendulum is a mechanical system of astonishing richness;
- its riches lie, for the most part, beyond the reach of classical analysis, but
- yield readily to numerical exploration;
- similar remarks pertain to almost *all* nonlinear mechanical systems.

It is a pleasure to acknowledge my debt to Joel Franklin for expert assistance in developing some of the computational strategies that are described in the text.