Probability density for partitions of n with k parts

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Recently N. A. Wheeler has posed questions in regard to the thermodynamics of the partition function $p(n, k)$, being the number of partitions of n into exactly k parts. Experimental plots of $p(n, k)$ for fixed n and $k \in [1, n]$ appear on the face of it to be "Maxwellian" (perhaps "Planckian") in the sense of a rising graph with a clear maximum, then a (supposedly) exponential tail for large k approaching n. Denoting standardly the celebrated partition count $p(n)$ of all partitions of n, one might conjecture that the probabilities

$$f_{n,k} := \frac{p(n,k)}{p(n)},$$

which of course satisfy the normalization

$$\sum_{k=1}^{n} f_{n,k} \ = \ 1,$$

represent something like a contour of Maxwellian speeds at a particular temperature.

This suppositions of thermodynamical or quantal contour may be approximately true in some local fashion, but overall such suppositions are false. There is a doubly-exponential distribution result of Erdös and Lehner on partition theory [1], which refers to $P(n, k)$ being the number of partitions of n having *at most* k parts. (Thus, formally, $p(n, k) = P(n, k) - P(n, k-1)$ for $n \geq 1$, and $P(n, 0) := 0$.) The Erdös–Lehner result is that for the assignment

$$X(k) := \frac{k}{\sqrt{n}} - \frac{1}{c}\log n,$$

we have

$$\lim_{n\to\infty} \frac{P(n,k)}{p(n)} = e^{-\frac{c}{2}e^{-\frac{2}{c}X(k)}}.$$

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Here the absolute constant is

$$c \;:=\; \pi\sqrt{\frac{2}{3}}.$$

This theorem leads to an asymptotic estimate for the probability density as a discrete form

$$f_{n,k} \;\sim\; e^{-\frac{c}{2}e^{-\frac{2}{c}X(k)}} - e^{-\frac{c}{2}e^{-\frac{2}{c}X(k-1)}}.$$

One might also replace the discrete differencing from $k \to k-1$ with a derivative, to obtain heuristically a *continuous* analogue (which we have also normalized with a prefactor depending on n):

$$f(n,k) \;=\; \frac{1}{1-e^{-\frac{2}{c}\sqrt{n}}} e^{-\frac{c}{2}\frac{k}{\sqrt{n}}}\; e^{-\frac{2}{c}\sqrt{n}\; e^{-\frac{c}{2}\frac{k}{\sqrt{n}}}}.$$

Here the variable k is now continuous, ranging as $k \in [0,\infty]$. (Though $k \le n$ in the discrete theory, it is convenient to allow any such k for this continuous density.) With these caveats we have, exact normalization

$$\int_0^\infty f(n,k)\; dk \;=\; 1.$$

It is remarkable that actual numerical plots show the continuous analogue to be inferior to the discrete form for relatively small k, as discovered by N. Wheeler.

Using the (suspect) continuous density—again heuristically—a maximum of said density should occur at the mode value

$$k_0 \;\sim\; \frac{1}{\pi}\sqrt{\frac{3}{2}}\,\sqrt{n}\;\log n.$$

This supposition turns out to be rigorously valid, in the sense that G. Szekeres established in 1953 [2] the sharper, yet consistent asymptotic

$$k_0 \;=\; \frac{\sqrt{6}}{\pi}\sqrt{n}\;L + \frac{6}{\pi^2}(3(L+1)/2 - L^2/4) - 1/2 + O((\log^4 n)/\sqrt{n}),$$

with $L := \log((1/\pi)\sqrt{6n})$.

It was also known to Erdös and Lehner that for $k << \sqrt{n}$ we have

$$P(n,k) \;\approx\; \frac{1}{k!}\binom{n-1}{k-1}.$$

From this perhaps one can use $p(n,k) = P(n,k) - P(n,k-1)$ to determine the small-k density $f_{n,k}$. It would be good to develop a unified theory of how such a small-k estimate joins with the doubly-exponential behavior for larger k.

References

[1] P. Erdos and J. Lehner , "The distribution of the number of summands in the partitions of a positive integer," Duke Math. J. 8 (1941), 335–345.

[2] G. Szekeres, "Some asymptotic formulae in the theory of partitions (II)," Quart. J. of Math. (Oxford), 4 (1953), 96-111,