Selected topics in

Quantum Measurement Theory

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Introduction. In Chapter 0 of my Quantum II Notes (Spring 2009) I wrote under the head **Rudiments of the quantum theory of measurement** as follows:

"Though a variety of other—equivalent or generalized—modes of state-representation will emerge, we can, for starters, assert that

the momentary *state* of a quantum system $\mathfrak{S}$
can be represented by a unit vector $|\psi)$ in $\mathcal{H}$

The specific identity of $\mathcal{H}$ is contingent ...upon general principles yet to be described, and upon the physical details of $\mathfrak{S}$.

"How, in such an abstract place as $\mathcal{H}$ do we secure ground to stand on? To what do we tie the thread that anchors us in experienced reality? Consider the corresponding classical question: how do we gain knowledge of the coordinates (q,p) the serve to describe the momentary state of a classical system? The answer, of course, is "by direct observation, by measurement." The situation in quantum mechanics is precisely the same, but with a difference: in classical physics the question is seldom posed/answered because measurement is considered classically to be conceptually straightforward, whatever may be the practical difficulties in particular cases. In quantum mechanics, on the other hand, the "measurement problem" is not at all trivial: it is central to the theory, is the source of much that is most characteristic of quantum physics and of conceptual issues that still after all these years remain profoundly surprising, sometimes baffling. But in its *mathematical* essentials of the quantum theory of measurement is quite simple (as any theory based on rudimentary linear algebra almost *has* to be).

"Quantum measurement theory springs from the theory of self-adjoint operators. Specifically, to every "classical observable"—*i.e.*, to every real-valued function $A(x,p)$ defined on classical phase space—we associate a self-adjoint linear operator $\mathbf{A}$ which acts upon the elements of $\mathcal{H}$. We then associate

- the possible meter-readings which can result from A-measurement with the (necessarily real) *eigenvalues* of A;
- the possible quantum states immediately subsequent to such a measurement with the *eigenvectors* of A.

"Each observable contrives spectrally to erect its own individual 'orthogonal scaffold $\{|a)\}$ in the space of states.' How that abstract construction becomes tied to the physical scaffold (inertial frame with attendant calebrated hardware) that we have constructed in the laboratory hinges upon our answer to this fundamental question:

> By what specific **rule of correspondence** is the association
>
> $$A(x,p) \longleftrightarrow \mathsf{A}$$
>
> to be established?

This is a question to which we will return. But for the moment...

"Look more closely to the idealized measurement process to which I have alluded. System $\mathfrak{S}$, in unknown quantum state $|\psi)$, is presented to (meaning 'brought into interaction with') the measurement device represented by the operator A (I will call such a device an 'A-meter'). After the interaction is complete

- the device is in the state a reported by its read-out mechanism, and this is interpreted to mean that
- the system $\mathfrak{S}$ is in—has by the meter been *placed* in—state $|a)$.

"Quantum mechanically fundamental is the fact that *repetitions yield statistically scattered results*: we obtain

$$|\psi) \xrightarrow[A\text{-measurement}]{\hspace{7em}} \begin{cases} |a_1) \text{ with probability } P_1 = |(a_1|\psi)|^2 \\ |a_2) \text{ with probability } P_2 = |(a_2|\psi)|^2 \\ \vdots \\ |a_n) \text{ with probability } P_n = |(a_n|\psi)|^2 \\ \vdots \end{cases} \tag{1.0}$$

Quantum measurement is by this scheme a 'state-preparation process,' and measurement devices are, in effect, sieves: the input state $|\psi)$ can, as we have seen, be resolved

$$|\psi) = \sum_i |a_i)(a_i|\psi)$$

and the device acts (probabilistically—*there's the rub*!) to

- to pass one of the eigen-components, and
- to annihilate all others.

We assert that a measurement has actually taken place on these grounds: if the output $|a_n)$ of a measurement which registered a_n is immediately re-presented to an A-meter we have (in automatic consequence of the scheme just described)

$$|a_n) \xrightarrow[\text{repeated } A\text{-measurement}]{} \begin{cases} |a_1) \text{ with probability } P_1 = |(a_1|a_n)|^2 = 0 \\ |a_2) \text{ with probability } P_2 = |(a_2|a_n)|^2 = 0 \\ \quad\vdots \\ |a_n) \text{ with probability } P_n = |(a_n|a_n)|^2 = 1 \\ \quad\vdots \end{cases}$$

which is to say: we recover (or 'confirm') the previous result with certainty.

"The expected average of many independent A-measurements (*i.e.*, of the results obtained when many identical copies of $|\psi)$ are presented serially to an A-meter) can be described

$$\begin{aligned} \langle a\rangle_\psi &\equiv \textstyle\sum_i a_i P_i \\ &= \textstyle\sum_i a_i |(a_i|\psi)|^2 \\ &= (\psi|\Big\{\textstyle\sum_i |a_i)a_i(a_i|\Big\}|\psi \\ &= (\psi|\mathbf{A}|\psi) \end{aligned} \tag{1.1}$$

but alernative descriptions exist and are sometimes more useful. For example, let $\{|n)\}$ be some arbitrary orthonormal basis in the space of states. Drawing upon the completeness condition (1.2), we have

$$\begin{aligned} &= \textstyle\sum_n (\psi|n)(n|\mathbf{A}|\psi) \\ &= \textstyle\sum_n (n|\mathbf{A}|\psi)(\psi|n) \\ &= \textstyle\sum_n (n|\mathbf{A}\boldsymbol{\rho}_\psi|n) \quad \text{where } \boldsymbol{\rho}_\psi \equiv |\psi)(\psi| \text{ projects onto } |\psi) \\ &= \operatorname{tr}\mathbf{A}\boldsymbol{\rho}_\psi \end{aligned} \tag{1.2}$$

In $\boldsymbol{\rho}_\psi$ we have encountered the germ of an idea that will grow up to become the 'density matrix,' which plays an indispensable role in a broad assortment of applications. The m^{th} moment of the measured data can be described variously

$$\begin{aligned} \langle a^m\rangle_\psi &\equiv \textstyle\sum_i (a_i)^m P_i \\ &= (\psi|\mathbf{A}^m|\psi) \\ &= \operatorname{tr}\big\{\mathbf{A}^m\boldsymbol{\rho}_\psi\big\} \end{aligned}$$

where use has been made of $\mathbf{A}^m = \sum_i |a_i)\,a_i^m(a_i|$. In the case $m = 0$ we have (for *any* observable)

$$\begin{aligned} \langle a^0\rangle_\psi &= \textstyle\sum_i P_i = 1 \quad &&: \quad \text{probabilities sum to unity} \\ &= (\psi|\psi) &&: \quad \text{state vector is normalized} \\ &= \operatorname{tr}\boldsymbol{\rho}_\psi \end{aligned} \tag{1.4}$$

"Quantum mechanics attempts to describe not 'where the next particle will land on the detection screen' but *statistical features of the pattern* formed when many identically-prepared particles are directed at the screen. And that it manages to do impressively well. Statements of the form (6) are standardly held to exhaust the physical output—the predictive power—of quantum theory. It was for this reason that Einstein (and Schrödinger too; also DeBroglie) considered quantum mechanics to 'nice so far as it goes, but obviously *incomplete*.'"

The remarks quoted above serve well enough to describe the bare-bones "projective theory of quantum measurement," a subject rooted in **von Neumann's projection postulate**.[1] It is by appeal to that theory that we analyze (for example) Bell's experiment in its many variations, the logistics of teleportation and other issues that touch experimentally on the foundations of quantum mechanics. The theory is easy to use, and entirely adequate to most work-a-day applications. But it is axiomatically disjoint from quantum *dynamics*, provides no insight into *how it happens* that quantum measurements come to be projective and (it is imagined) instantaneous, no insight into *why* are "classical," themselves semi-exempt from the laws of quantum mechanics.

Any attempt to account for the mechanics of measurement processes by appeal to the principles of quantum dynamics may, of course, be doomed from the outset, since quantum dynamics is deterministic/unitary/reversible while quantum measurements are held to be nondeterministic/irreversible. Which is what leads Yamamoto & İmamağlu to state without qualification that "The standard quantum theory based on the Schrödinger equation cannot describe the process of measurement." They remark that "Interest among experimental physicists shifted approximately 20 years ago from one-time destructive measurements on an ensemble of quantum objects to repeated non-destructive measurements on a single quantum object," and—setting aside the "how it happens" problem—undertake to describe some of those modern refinements/ extensions of the von Neumann scheme.

Which will be my objective too. Working from Yamamoto & İmamağlu, §2.4 in Breuer & Petruccione[2] and certain web sites[3] I want to clarify my understanding of (for example) the distinction between demolition and non-demolition, direct and indirect measurements, the meaning of "back-action," what people mean when they speak of the "simultaneous measurement of conjugate observables, *etc*.

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[1] See §1.3 Yoshihisa Yamamoto & Ataç İmamağlu's *Mesoscopic Quantum Optics* (1999). Their Chapter I provides a good introduction to aspects of the quantum theory of measurement that lie *beyond* von Neumann's projection postulate, and that acquire importance when one looks to the quantum theory of open systems, to the theory of decoherence, or to the theoretical foundations of some recent experimental activity.

[2] H.-P. Breuer & F. Petruccione, *The Theory of Open Quantum Systems* (2006).

[3] See, for example, http://en.wikipedia.org/wiki/Measurement_in_quantum_mechanics.

Refined projection postulate. The refinements I have in mind come into view when elects to use a density operator $\boldsymbol{\rho}$ to describe the state of $\mathfrak{S}$, and proceed from the spectral representation

$$\mathsf{A} = \sum_m a_m \mathsf{P}_m \tag{2}$$

of the self-adjoint operator A that describes the postulated action of an A-meter. Here the eigenvalues a_m are understood to be distinct, but μ_m-fold degenerate The self-adjoint operator P_m projects onto the μ_m-dimensional m^{th} eigenspace. One has

$$\begin{aligned} \mathsf{P}_m \mathsf{P}_m &= \mathsf{P}_m \\ \mathsf{P}_m \mathsf{P}_n &= \mathsf{0} \quad : \quad m \neq n \\ \textstyle\sum_m \mathsf{P}_m &= \mathsf{I} \end{aligned}$$

The P_m are tracewise orthogonal:

$$\text{tr}\big\{\mathsf{P}_m \mathsf{P}_n\big\} = 0 \quad : \quad m \neq n$$

Moreover,

$$\text{tr}\big\{\mathsf{P}_m\big\} = \mu_m$$

We can fold those two statements into a single statement

$$(\mathsf{P}_m, \mathsf{P}_n) \equiv \text{tr}\big\{\mathsf{P}_m \mathsf{P}_n\big\} = g_{mn} \quad : \quad g_{mn} = \mu_m \delta_{mn}$$

It now follows that if

$$\mathsf{B} = \textstyle\sum_k \mathsf{P}_k\, b^k \tag{3.1}$$

then

$$(\mathsf{P}_n, \mathsf{B}) = g_{nk} b^k \quad \Longrightarrow \quad b^m = g^{mn}(\mathsf{P}_n, \mathsf{B})$$

where the g^{mn} are elements of $\|g_{mn}\|^{-1}$: $g^{mn} = (\mu_m)^{-1}\delta^{mn}$. So we have this "Fourier identity"

$$\begin{aligned} \mathsf{B} = \textstyle\sum_{mn} \mathsf{P}_m (\mu_m)^{-1} \delta^{mn} (\mathsf{P}_n, \mathsf{B}) &= \textstyle\sum_m \mathsf{P}_m (\mu_m)^{-1} (\mathsf{P}_m, \mathsf{B}) \\ &= \textstyle\sum_m \mathsf{P}_m \frac{\text{tr}\{\mathsf{P}_m \mathsf{B}\}}{\text{tr}\{\mathsf{P}_m\}} \end{aligned} \tag{3.2}$$

As a test of the correctness of this result we take the trace of the expression on the right and obtain $\sum_m \text{tr}\big\{\mathsf{P}_m \mathsf{B}\big\} = \text{tr}\big\{\mathsf{B}\big\}$.

It is important to notice the respect in which the spectral representation (2) of the self-adjoint operator A is special. To develop the point I have now in mind, I find it convient to distinguish two cases:

$\boxed{\text{NON-DEGENERATE } \mathsf{A} \text{ SPECTRUM}}$ Equation (2) is now simply an elegant way of writing

$$\mathsf{A} = \sum_m |a_m)a_m(a_m|$$

where the operators $\mathsf{P}_m \equiv |a_m)(a_m|$ project now onto orthogonal 1-spaces. An *arbitrary* self-adjoint operator H can be developed in the A-basis

$$\begin{aligned}\mathsf{H} &= \sum_{m,n} |a_m)(a_m|\mathsf{H}|a_n)(a_n| \\ &= \sum_{m,n} H_{mn}\mathsf{P}_{mn} \quad \text{with} \quad \mathsf{P}_{mn} \equiv |a_m)(a_n|\end{aligned}$$

but the operators P_{mn} are *projective if and only if* $m = n$. Equations of the form (3) are therefore special to cases in which all off-diagonal terms vanish: $H_{mn} = h_m\delta_{mn}$, and in the absence of spectral degeneracy those equations simplify: we then have

$$(\mathsf{P}_m, \mathsf{P}_n) \equiv \text{tr}\big\{\mathsf{P}_m\mathsf{P}_n\big\} = \delta_{mn}$$

$$\mathsf{B} = \textstyle\sum_m \mathsf{P}_m \text{tr}\big\{\mathsf{P}_m\mathsf{B}\big\}$$

The projection operators P_m then serve to chop state vectors $|\psi)$ into orthogonal fragments:

$$|\psi) = \sum_m |\psi_m) \quad \text{with} \quad |\psi_m) \equiv \mathsf{P}_m|\psi)$$

In non-degenerate cases the projection postulate is (see again (1.0)) standardly understood to state

$$\begin{aligned}&|\psi) = \textstyle\sum_m |a_m)(a_m|\psi) \\ &\downarrow \\ &|a_n) \quad \text{with probability } P_n = \|(a_n|\psi)\|^2 = (\psi|\mathsf{P}_n|\psi)\end{aligned}$$

which in density operator language becomes

$$\begin{aligned}\boldsymbol{\rho}_\psi \equiv |\psi)(\psi| &= \textstyle\sum_{mn} |a_m)(a_m|\psi)(\psi|a_n)(a_n| \\ &\downarrow \quad \text{“premeasurement”: abandonment of off-diagonal terms} \\ \boldsymbol{\rho}_{\text{intermediate}} &= \textstyle\sum_m |a_m)(a_m|\psi)(\psi|a_m)(a_m| \\ &= \textstyle\sum_m \mathsf{P}_m \,\text{tr}\big\{\mathsf{P}_m\boldsymbol{\rho}_\psi\big\} \\ &\downarrow \quad \text{projective measurement} \\ \boldsymbol{\rho}_{\text{after}} &= \mathsf{P}_n \quad \text{with probability } P_n = \text{tr}\big\{\mathsf{P}_n\boldsymbol{\rho}_\psi\big\}\end{aligned} \tag{4}$$

$\boxed{\text{DEGENERATE } \mathsf{A} \text{ SPECTRUM}}$ We can in such cases speak usefully only about H-operators of (what we now recognize to be) the specialized structure

(3.1). The projection operators P_m then serve as before to chop state vectors $|\psi)$ into orthogonal fragments:

$$|\psi) = \sum_m |\psi_m) \quad \text{with} \quad |\psi_m) \equiv \mathsf{P}_m|\psi)$$

but $|\psi_m)$ refers now to a designated one of the vectors in μ_m-dimensional m^{th} eigenspace of $\mathbf{A}$. In degenerate cases the projection postulate is understood to state

$$\begin{aligned}
&|\psi) = \textstyle\sum_m |\psi_m)\\
&\qquad\downarrow\\
&\frac{|\psi_n)}{\sqrt{(\psi_n|\psi_n)}} \quad \text{with probability } P_n = |(\psi_n|\psi_n)|^2 = (\psi|\mathsf{P}_n|\psi)
\end{aligned}$$

which in density operator language becomes

$$\begin{aligned}
\boldsymbol{\rho}_\psi \equiv |\psi)(\psi| &= \textstyle\sum_{mn} \mathsf{P}_m|\psi)(\psi|\mathsf{P}_n\\
&\downarrow \quad \text{“premeasurement”: abandonment of off-diagonal terms}\\
\boldsymbol{\rho}_{\text{intermediate}} &= \textstyle\sum_m |a_m)(a_m|\psi)(\psi|a_m)(a_m|\\
&= \textstyle\sum_m \mathsf{P}_m \operatorname{tr}\big\{\mathsf{P}_m\,\boldsymbol{\rho}_\psi\big\} \qquad (5)\\
&\downarrow \quad \text{projective measurement}\\
\boldsymbol{\rho}_{\text{after}} &= \frac{\mathsf{P}_n\,\boldsymbol{\rho}_\psi\,\mathsf{P}_n}{\operatorname{tr}\big\{\mathsf{P}_n\,\boldsymbol{\rho}_\psi\,\mathsf{P}_n\big\}} \quad \text{with probability } P_n = \operatorname{tr}\big\{\mathsf{P}_n\,\boldsymbol{\rho}_\psi\,\mathsf{P}_n\big\}
\end{aligned}$$

The adjustment $|\psi)(\psi| \longmapsto \sum_\alpha p_\alpha|\psi_\alpha)(\psi_\alpha|$ does not compromise the validity of the preceding statements, so they pertain as well to cases in which the initial density matrix $\boldsymbol{\rho}$ refers to a mixed state. Notice that

$$\begin{aligned}
\textstyle\sum_n P_n &= \operatorname{tr}\big\{\textstyle\sum_n \mathsf{P}_n\,\boldsymbol{\rho}\,\mathsf{P}_n\big\}\\
&= \operatorname{tr}\big\{\textstyle\sum_n \mathsf{P}_n\,\mathsf{P}_n\,\boldsymbol{\rho}\big\}\\
&= \operatorname{tr}\big\{\textstyle\sum_n \mathsf{P}_n\,\boldsymbol{\rho}\big\} \quad \text{by projectivity}\\
&= \operatorname{tr}\big\{\boldsymbol{\rho}\big\} \quad \text{by completeness}\\
&= 1
\end{aligned}$$

which is gratifying. Trivially $\operatorname{tr}\boldsymbol{\rho}_{\text{after}} = 1$ but the evaluation of $\operatorname{tr}\boldsymbol{\rho}^2_{\text{after}}$ is nontrivial. We have

$$\operatorname{tr}\boldsymbol{\rho}^2_{\text{after}} = \frac{\operatorname{tr}\big\{(\mathsf{P}_n\,\boldsymbol{\rho})^2\big\}}{\operatorname{tr}\big\{\mathsf{P}_n\,\boldsymbol{\rho}\big\}^2}$$

I am satisfied on the basis of numerical experiments (but am not presently in position to prove) that[4]

$$1 \geqslant \frac{\operatorname{tr}\big\{(\mathsf{P}\,\boldsymbol{\rho})^2\big\}}{\operatorname{tr}\big\{\mathsf{P}\,\boldsymbol{\rho}\big\}^2} \geqslant \operatorname{tr}\big\{\boldsymbol{\rho}^2\big\} \quad : \quad \text{all density operators } \boldsymbol{\rho}, \text{ all projectors } \mathsf{P}$$

[4] NOTE: In subsequently extended experiments I did encounter a single instance of a case in which the second inequality was violated.

and on that basis assert that

- measurement of pure states yields pure states;
- measurement of mixed states yields states that are purer than they were.

John von Neumann—who invented (simultaneously with Hermann Weyl) the density operator—laid down the foundations of this subject in Chapters V & VI of *Mathematische Grundlaen der Quantenmechanik* (1932), which are reprinted (Robert Beyer translation, 1955) in J. A. Wheeler & W. H. Zurek, *Quantum Theory and Measurement* (1983). In a footnote (their page 550) Wheeler & Zurek thank A. S. Wightman for pointing out that von Neumann lapsed into error when he tried to discuss the case in which the A-spectrum is degenerate, and cite G. Lüders ("Über die Zustandsänderung den Messprozess," Ann. Physik **8**, 322–328 (1951)) and a paper by W. H. Furry (1966) for corrected treatment of such cases. It is for this reason that one sometimes encounters reference[3] to the "von Neumann-Lüders projection postulate."

Alice and Bob (again). Suppose the system of interest to Alice is a fragment $\mathfrak{S}_A$ of the composite system $\mathfrak{S}_A \oplus \mathfrak{S}_B$, and that Bob has interest in (and access to) only the other fragment. In such a setting, Alice's a-meter acquires (in God's view) the spectral representation

$$\mathsf{A} = \sum_m a_m \big(\mathsf{p}_m \otimes \mathsf{I}_B\big) \equiv \sum_m a_m \mathsf{P}_m$$

while Bob's b-meter is represented[5]

$$\mathsf{B} = \sum_n b_n \big(\mathsf{I}_A \otimes \mathsf{q}_n\big) \equiv \sum_n b_n \mathsf{Q}_n$$

Notice that even if Alice's a-meter were spectrally non-degenerate, the eigenvalues of A would display a degeneracy equal to the dimension of I_B. So when one looks to composite systems the degenerate case—which in the preceding discussion seemed exceptional—becomes the only game in town.

Suppose the composite system to be initially in state $\boldsymbol{\rho}$. Alice makes a measurement, gets a_m with probability

$$P_{\text{Alice}}(a_m) = \operatorname{tr}\big\{\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m\big\} \tag{6.1}$$

and leaves the system in the post-measurement state

$$\boldsymbol{\rho}_{\text{after Alice}} = \frac{\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m}{\operatorname{tr}\big\{\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m\big\}} \tag{6.2}$$

Now Bob makes a measurement, gets b_n with conditional probability

$$\begin{aligned} P(b_n|a_m) &= \operatorname{tr}\big\{\mathsf{Q}_n\,\boldsymbol{\rho}_{\text{after Alice}}\,\mathsf{Q}_n\big\} \\ &= \frac{\operatorname{tr}\big\{\mathsf{Q}_n\,\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m\,\mathsf{Q}_n\big\}}{\operatorname{tr}\big\{\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m\big\}} \end{aligned} \tag{6.3}$$

[5] I will henceforth omit the subscripts from I_A and I_B, since they are always clear from context.

and creates the state

$$\begin{aligned}\boldsymbol{\rho}_{\text{after Alice/Bob}} &= \frac{\mathsf{Q}_n\,\boldsymbol{\rho}_{\text{after Alice}}\,\mathsf{Q}_n}{\text{tr}\big\{\mathsf{Q}_n\,\boldsymbol{\rho}_{\text{after Alice}}\,\mathsf{Q}_n\big\}}\\ &= \frac{\mathsf{Q}_n\,\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m\,\mathsf{Q}_n}{\text{tr}\big\{\mathsf{Q}_n\,\mathsf{P}_m\,\boldsymbol{\rho}\,\mathsf{P}_m\,\mathsf{Q}_n\big\}}\end{aligned} \tag{6.4}$$

If, on the other hand, Bob had made his measurement *before* Alice made hers we would have[6]

$$P_{\text{Bob}}(b_n) = \text{tr}\big\{\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\big\} \tag{7.1}$$

$$\boldsymbol{\rho}_{\text{after Bob}} = \frac{\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n}{\text{tr}\big\{\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\big\}} \tag{7.2}$$

and

$$\begin{aligned}P(a_m|b_n) &= \text{tr}\big\{\mathsf{P}_m\,\boldsymbol{\rho}_{\text{after Bob}}\,\mathsf{P}_m\big\}\\ &= \frac{\text{tr}\big\{\mathsf{P}_m\,\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\,\mathsf{P}_m\big\}}{\text{tr}\big\{\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\big\}}\\ &= \frac{P_{\text{Alice}}(a_m)}{P_{\text{Bob}}(b_n)}\cdot P(b_n|a_m)\end{aligned} \tag{8.3}$$

producing the final state

$$\boldsymbol{\rho}_{\text{after Bob/Allice}} = \frac{\mathsf{P}_m\,\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\,\mathsf{P}_m}{\text{tr}\big\{\mathsf{P}_m\,\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\,\mathsf{P}_m\big\}} \tag{8.4}$$

All P projectors commute with all Q projectors,[7] so this is the same final state as we obtained before:

$$= \boldsymbol{\rho}_{\text{after Bob/Allice}} \tag{9}$$

Generally, $P(a,b) = P(a|b)P(b) = P(b|a)P(a)$, which supply

$$P(a|b) = \frac{P(a)}{P(b)}P(b|a)$$

of which (7.3) provides an instance. We conclude that all the statistical results reported above can be considered to follow from the joint distribution

$$P_{mn} \equiv P(a_m, b_n) = \text{tr}\big\{\mathsf{P}_m\,\mathsf{Q}_n\,\boldsymbol{\rho}\,\mathsf{Q}_n\,\mathsf{P}_m\big\} \tag{10}$$

Immediately

$$\sum_{m,n} P(a_m, b_n) = \text{tr}\,\boldsymbol{\rho} = 1$$

as required.

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[6] No confusion can result if we drop the identifying subscripts from $P_{\text{Alice}}(a_m)$ and $P_{\text{Bob}}(b_n)$, which henceforth I will do.

[7] We have $[\mathsf{A}, \mathsf{B}] = \mathbf{0}$ even when $[\mathsf{a}, \mathsf{b}] \neq \mathbf{0}$.

EXAMPLE: Alice and Bob are equipped with identical instruments

$$\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which in God's view are represented

$$\mathbb{A} = \boldsymbol{\sigma}_3 \otimes \mathbb{I}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbb{B} = \mathbb{I}_2 \otimes \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that the obvious eigenvalues ± 1 of $\boldsymbol{\sigma}_3$ are non-degenerate, while the eigenvalues of $\mathbb{A}$ and $\mathbb{B}$ are doubly degenerate. It is obvious also that

$$\mathbb{A} = \mathbb{P}_1 - \mathbb{P}_2 \quad \text{with} \quad \mathbb{P}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{B} = \mathbb{Q}_1 - \mathbb{Q}_2 \quad \text{with} \quad \mathbb{Q}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{Q}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and that those projectors project onto 2-spaces. Suppose, moreover, that their composite pair of qubits is initially in the Bell state

$$\psi = \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

which can be represented

$$\boldsymbol{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Working now from (10)—which for computational purposes is more efficiently written

$$P_{mn} = \text{tr}\big\{\mathbb{P}_m \mathbb{Q}_n \,\boldsymbol{\rho}\,\big\}$$

—we find

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

which provides vivid indication that Alice/Bob's meter readings are perfectly anticorrelated. That observation acquires quantitative from Karl Pearson's correlation coefficient

$$\begin{aligned} r &= \frac{\langle (a - \langle a \rangle)(b - \langle b \rangle) \rangle}{\sqrt{\langle (a - \langle a \rangle)^2}\sqrt{\langle (b - \langle b \rangle)^2}} \\ &= \frac{\langle ab \rangle - \langle a \rangle \langle b \rangle}{\sqrt{\langle a^2 \rangle - \langle a \rangle^2}\sqrt{\langle b^2 \rangle - \langle b \rangle^2}} \end{aligned}$$

where the expected values of the random variables a and b—the values of which range on ±1—are computed by appeal to $P(a,b)$ and to the implied marginal probabilities

$$\begin{aligned} p(a_1) &= P(a_1,b_1) + P(a_1,b_2) = \tfrac{1}{2}\\ p(a_2) &= P(a_2,b_1) + P(a_2,b_2) = \tfrac{1}{2}\\ p(b_1) &= P(a_1,b_1) + P(a_2,b_1) = \tfrac{1}{2}\\ p(b_2) &= P(a_1,b_2) + P(a_2,b_2) = \tfrac{1}{2} \end{aligned} \qquad (\star)$$

Specifically, we find

$$\begin{aligned} \langle a\ \rangle &= \langle b\ \rangle = 0\\ \langle a^2\rangle &= \langle b^2\rangle = 1\\ \langle ab\rangle &= -1 \end{aligned}$$

whence

$$r = -1$$

which signals perfect anticorrelation. "Alice's effective density matrix ρ_{Alice}" is obtained by "tracing out Bob's part" of ρ:

$$\rho_{\text{Alice}} = \text{tr}_2\,\rho = \begin{pmatrix} \frac{1}{2} & 0\\ 0 & \frac{1}{2} \end{pmatrix}$$

Drawing upon the spectral resolution of her meter

$$\sigma_3 = (+1)\begin{pmatrix} 1 & 0\\ 0 & 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 & 0\\ 0 & 1 \end{pmatrix}$$

Alice computes

$$\begin{aligned} p(a_1) &= \text{tr}\Big\{\begin{pmatrix} 1 & 0\\ 0 & 0 \end{pmatrix}\rho_{\text{Alice}}\Big\} = \tfrac{1}{2}\\ p(a_2) &= \text{tr}\Big\{\begin{pmatrix} 0 & 0\\ 0 & 1 \end{pmatrix}\rho_{\text{Alice}}\Big\} = \tfrac{1}{2} \end{aligned}$$

in precise agreement with results quoted at $(\star)$. Bob obtains similar resuilts by similar means. But results computed in this way provide no indication that if Alice and Bob were to compare their data they would discover perfect anticorrelation: from $(\star)$ it is not possible to recover $P(a,b)$: all tables

$$\begin{pmatrix} P_{11}+c & P_{12}-c\\ P_{21}-c & P_{22}+c \end{pmatrix}$$

lead to the same set of marginal distributions.[8] The initial state was pure

$$\text{tr}\,\rho = \text{tr}\,\rho^2 = 1$$

---

[8] The requirement that all elements of the table be non-negative serves to constrain the admissible values of c.

but entangled

$$\rho_{\text{Alice}} \otimes \rho_{\text{Bob}} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \neq \rho$$

What can we say in these respects about the state of the composite system after Alice/Bob have completed their measurements? Working from (6.4)

$$\rho_{\text{after Alice/Bob}:m,n} = \frac{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n}{\mathrm{tr}\big\{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n\big\}}$$

we find that $\rho_{\text{after Alice/Bob}:m,n}$ becomes indeterminate in the "impossible cases" —the cases that occur with zero probability because

$$\mathrm{tr}\big\{\mathbb{Q}_1 \mathbb{P}_1 \, \rho \, \mathbb{P}_1 \mathbb{Q}_1\big\} = \mathrm{tr}\big\{\mathbb{Q}_2 \mathbb{P}_2 \, \rho \, \mathbb{P}_2 \mathbb{Q}_2\big\} = 0$$

Looking only, therefore, to the possible cases, we find

$$\rho_{\text{after Alice/Bob}:1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_{\text{after Alice/Bob}:2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

both of which are found to be pure (as was seen at the top of page 8 to follow from the fact that the initial state ρ was pure) and—more remarkably—*disentangled*. I have assigned ρ a great number of values (each of which referred typically to an entangled mixed state) and found the equation

$$\mathrm{tr}_2\bigg[\frac{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n}{\mathrm{tr}\big\{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n\big\}}\bigg] \otimes \mathrm{tr}_2\bigg[\frac{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n}{\mathrm{tr}\big\{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n\big\}}\bigg] = \frac{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n}{\mathrm{tr}\big\{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n\big\}}$$

to be valid in all instances, even when I assigned to m, n the previously forbidden values,[9] and even when I supplied Alice/Bob with distinct arbitrarily constructed meters represented by matrices of (generally) distinct dimension. **Disentanglement by measurement** appears on this evidence to be a general phenomenon. I found, moreover, that if the spectra of $\mathbb{P}_m$ and $\mathbb{Q}_n$ are non-degenerate ; *i.e.*, if they project onto one-spaces, can be described

$$\begin{aligned} \mathbb{P}_m &= \boldsymbol{a}_m \boldsymbol{a}_m^+ \\ \mathbb{Q}_n &= \boldsymbol{b}_n \, \boldsymbol{b}_n^+ \end{aligned}$$

[9] In randomly constructed cases one cannot expect to encounter

$$\mathrm{tr}\big\{\mathbb{Q}_n \mathbb{P}_m \, \rho \, \mathbb{P}_m \mathbb{Q}_n\big\} = 0$$

whatever the values assigned to m, n.

then one also has **purification by measurement**. Which is, in fact, not that surprising: the vector

$$\boldsymbol{f}_{mn} \equiv \boldsymbol{a}_m \otimes \boldsymbol{b}_n$$

clearly refers to a disentangled (separable) pure state, and one finds that

$$\mathbb{F}_{mn} \equiv \frac{\mathbb{Q}_n \mathbb{P}_m\, \not{\rho}\, \mathbb{P}_m \mathbb{Q}_n}{\mathrm{tr}\big\{\mathbb{Q}_n \mathbb{P}_m\, \not{\rho}\, \mathbb{P}_m \mathbb{Q}_n\big\}} = \boldsymbol{f}_{mn} \boldsymbol{f}_{mn}^{+}$$

That construction fails, however, if the spectra of either/both of the projectors $\mathbb{P}_m$, $\mathbb{Q}_n$ is degenerate (*i.e.*, if either projects onto a space of dimension greater than one). One would like to establish these points by analytical argument (rather than by computational experimentation), but I will save that exercise for another day.

I conclude that (10)

$$P(a_m, b_n) = \mathrm{tr}\big\{\, \mathsf{P}_m \mathsf{Q}_n\, \boldsymbol{\rho}\, \mathsf{Q}_n \mathsf{P}_m \big\}$$

provides a maximally efficient basis for the analysis of "Alice/Bob experiments" of all types—Bohm's version of EPR, Bell's experiment and all variants thereof. And that it generalizes straightforwardly to include (for example) experiments in which also Chris is a player. I conclude also that $\boldsymbol{\rho}$-formalism is vastly more powerful and illuminating that the (less general) $|\psi)$-formalism—the formalism of choice in all fundamental contexts.

Motion of a reduced density operator. Any effort to look behind the projection postulate to the quantum dynamics of the measurement process requires one to take into explicit account the fact that system $\mathcal{S}$ and meter $\mathcal{M}$ are components of a composite system $\mathcal{S} \oplus \mathcal{M}$, the state of which is represented at each instant by a density operator $\boldsymbol{\rho}$. A composite Hamiltonian—which has generally the form

$$\mathsf{H} = \mathsf{H}_{\text{free system}} + \mathsf{H}_{\text{free meter}} + \mathsf{H}_{\text{interaction}}$$

generates the motion of the composite system

$$i\hbar \tfrac{\partial}{\partial t} \boldsymbol{\rho} = [\, \mathsf{H}, \boldsymbol{\rho}\,]$$
$$\Downarrow$$
$$\boldsymbol{\rho}_t = \mathsf{U}(t)\, \boldsymbol{\rho}_0\, \mathsf{U}^{+}(t)$$

where (when H is time-independent) $\mathsf{U}(t) = \exp\big\{ -\tfrac{i}{\hbar} \mathsf{H}\, t \big\}$. All the observable properties of $\mathcal{S}$ are latent in the structure of the reduced density operator

$$\boldsymbol{\rho}_s = \mathrm{tr}_m \boldsymbol{\rho} \equiv \sum_q \big(\mathbb{I} \otimes (f_q| \big)\, \boldsymbol{\rho}\, \big(\mathbb{I} \otimes |f_q) \big)$$

while those of $\mathcal{M}$ are latent in the structure of

$$\boldsymbol{\rho}_m = \mathrm{tr}_s \boldsymbol{\rho} \equiv \sum_q \big((e_p| \otimes \mathbb{I} \big)\, \boldsymbol{\rho}\, \big(|e_p) \otimes \mathbb{I} \big)$$

Here $\{|e_p)\}$ is an arbitrary orthonormal basis in $\mathcal{H}_s$, and $\{|f_q)\}$ is an arbitrary orthonormal basis in $\mathcal{H}_m$. The question before us: How does $\boldsymbol{\rho}_s$ move? If we can answer the question we will know also how $\boldsymbol{\rho}_m$ moves, and be better positioned to consider the dynamically-developed correlations which would appear to be the name of the game in quantum measurement theory.

The equations we seek are known in the literature as "master equations," and come in several flavors. Recently I constructed a discussion of this subject that borrowed heavily from the following sources: Maximillian Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (2007), especially Chapter 4 ("Master-equation formulations of decoherence"); Ulrich Weiss, *Quantum Dissipative Systems* (3rd edition 2008), §2.3 and—my primary source —H.-P. Breier & F. Petriccione, *The Theory of Open Quantum Systems* (2006). What follows is a revision of that earlier material in which I draw also upon what I have learned by numerical simulation (see the *Mathematica* notebook "Motion of the reduced density operator" (12 June 2009)).

We work from

$$\boldsymbol{\rho}_{st} = \text{tr}_m\big\{\mathbf{U}(t)\boldsymbol{\rho}_0\,\mathbf{U}^{-1}(t)\big\} \tag{11}$$

and from the assumption that $\boldsymbol{\rho}_0$ is disentangled

$$\boldsymbol{\rho}_0 = \boldsymbol{\rho}_{s0}\otimes\boldsymbol{\rho}_{m0}$$

We assume moreover—quite unrealistically (!) in most contexts, but in service of our formal objective—that we possess the spectral resolution of $\boldsymbol{\rho}_{m0}$:

$$\boldsymbol{\rho}_{m0} = \sum_\alpha \lambda_\alpha|\phi_\alpha)(\phi_\alpha|$$

where the λ_α are positive real numbers that sum to unity. We now have

$$\boldsymbol{\rho}_{st} = \sum_q \big(\,\mathbf{I}\otimes(f_q|\,\big)\,\mathbf{U}\cdot\big(\boldsymbol{\rho}_{s0}\otimes\sum_\alpha\lambda_\alpha|\phi_\alpha)(\phi_\alpha|\,\big)\cdot\mathbf{U}^+\big(\,\mathbf{I}\otimes|f_q)\big)$$

Use

$$\boldsymbol{\rho}_{s0}\otimes\sum_\alpha\lambda_\alpha|\phi_\alpha)(\phi_\alpha| = \sum_\alpha\big(\,\mathbf{I}\otimes\sqrt{\lambda_\alpha}|\phi_\alpha)\big)\underbrace{\big(\boldsymbol{\rho}_{s0}\otimes 1\big)}_{\boldsymbol{\rho}_{s0}}\big(\,\mathbf{I}\otimes\sqrt{\lambda_\alpha}(\phi_\alpha|\,\big)$$

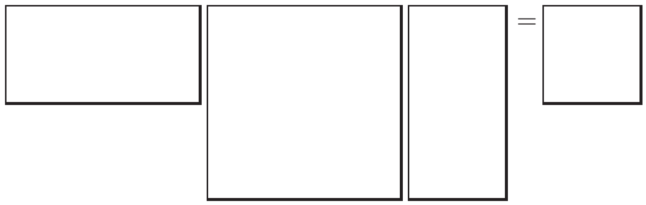
(here 1 is the one by one identity matrix) to obtain

$$\boldsymbol{\rho}_{st} = \sum_{\alpha,\beta}\mathbf{W}_{\alpha\beta}(t)\,\boldsymbol{\rho}_{s0}\,\mathbf{W}^+_{\alpha\beta}(t) \tag{12.1}$$

with

$$\left.\begin{aligned}\mathbf{W}_{\alpha\beta}(t) &\equiv \sqrt{\lambda_\alpha}\big[\mathbf{I}_s\otimes(\psi_\beta|\big]\,\mathbf{U}(t)\quad\big[\mathbf{I}_s\otimes|\phi_\alpha)\big]\\ \mathbf{W}^+_{\alpha\beta}(t) &\equiv \sqrt{\lambda_\alpha}\big[\mathbf{I}_s\otimes(\phi_\alpha|\big]\,\mathbf{U}^+(t)\big[\mathbf{I}_s\otimes|\psi_\beta)\big]\end{aligned}\right\} \tag{12.2}$$

The dimensions of the matrices that enter into the preceding definitions conform to the following pattern



where the short sides have length s, the long sides have length $s+m$. It now follows from $\sum_\beta |\psi_\beta)(\psi_\beta| = \mathsf{I}_m$, normality $(\phi_\alpha|\phi_\alpha) = 1$ and $\sum_\alpha \lambda_\alpha = 1$ that

$$\begin{aligned}
\sum_{\alpha,\beta} &\mathsf{W}^+_{\alpha\beta}(t)\,\mathsf{W}_{\alpha\beta}(t)\\
&= \sum_\alpha \lambda_\alpha \big[\mathsf{I}_s \otimes (\phi_\alpha|\big]\,\mathsf{U}^+(t)\big[\mathsf{I}_s \otimes \sum_\beta |\psi_\beta)(\psi_\beta|\big]\,\mathsf{U}(t)\big[\mathsf{I}_s \otimes |\phi_\alpha)\big]\\
&= \sum_\alpha \lambda_\alpha \big[\mathsf{I}_s \otimes (\phi_\alpha|\big]\,\mathsf{U}^+(t)\big[\mathsf{I}_s \otimes \mathsf{I}_m\big]\,\mathsf{U}(t)\big[\mathsf{I}_s \otimes |\phi_\alpha)\big]\\
&= \sum_\alpha \lambda_\alpha \big[\mathsf{I}_s \otimes (\phi_\alpha|\big]\,\mathsf{I}_{s+m}\big[\mathsf{I}_s \otimes |\phi_\alpha)\big]\\
&= \sum_\alpha \lambda_\alpha \big[\mathsf{I}_s \otimes (\phi_\alpha|\big]\big[\mathsf{I}_s \otimes |\phi_\alpha)\big] \quad \text{by } \mathsf{I}_s{\cdot}\mathsf{I}_s = \mathsf{I}_s \text{ and } \mathsf{I}_m|\phi_\alpha) = |\phi_\alpha)\\
&= \sum_\alpha \lambda_\alpha \big[\mathsf{I}_s \otimes 1\big]\\
&= \mathsf{I}_s
\end{aligned} \tag{13}$$

Returning with this information to (12.1), we have <u>conservation of probability so far as relates to the $\mathcal{S}$ system</u>

$$\mathrm{tr}\big\{\boldsymbol{\rho}_{st}\big\} = \mathrm{tr}\big\{\boldsymbol{\rho}_{s0}\cdot \sum_{\alpha,\beta} \mathsf{W}^+_{\alpha\beta}(t)\,\mathsf{W}_{\alpha\beta}(t)\big\} = \mathrm{tr}\big\{\boldsymbol{\rho}_{s0}\big\} = 1$$

but this critically important fact is actually a direct implication of (11): though in all cases $\mathrm{tr}\big\{\mathbb{A}\mathbb{B}\big\} = \mathrm{tr}\big\{\mathbb{B}\mathbb{A}\big\}$ it is generally the case that $\mathrm{tr}_s\big\{\mathbb{A}\mathbb{B}\big\} \neq \mathrm{tr}_s\big\{\mathbb{B}\mathbb{A}\big\}$ but invariably the case that $\mathrm{tr}\big\{\mathrm{tr}_s\big\{\mathbb{A}\mathbb{B}\big\}\big\} = \mathrm{tr}\big\{\mathrm{tr}_s\big\{\mathbb{B}\mathbb{A}\big\}\big\}$, which is sufficient to establish the point at issue.

Equations (12) do provide a description of $\boldsymbol{\rho}_{s0} \longmapsto \boldsymbol{\rho}_{st}$, but presume that we possess information—the spectral representation of $\boldsymbol{\rho}_{e0}$, an evaluation of $\mathsf{U}(t)$—that in realistic cases we cannot expect to have. It is, in this respect, gratifying to observe that in the absence of system-environmental interaction we have $\mathsf{U}(t) = \mathsf{U}_s(t) \otimes \mathsf{U}_m(t)$ and (12) reads

$$\begin{aligned}
\boldsymbol{\rho}_{st} &= \big[\mathsf{U}_s(t)\,\boldsymbol{\rho}_{s0}\,\mathsf{U}^+_s(t)\big] \otimes \sum_{\alpha,\beta} (\psi_\beta|\mathsf{U}_m(t)|\phi_\alpha)\lambda_\alpha(\phi_\alpha|\mathsf{U}^+_m(t)|\psi_\beta)\\
&= \big[\mathsf{U}_s(t)\,\boldsymbol{\rho}_{s0}\,\mathsf{U}^+_s(t)\big] \otimes \mathrm{tr}\big\{\mathsf{U}_m(t)\,\boldsymbol{\rho}_{e0}\,\mathsf{U}^+_m(t)\big\}\\
&= \ \mathsf{U}_s(t)\,\boldsymbol{\rho}_{s0}\,\mathsf{U}^+_s(t)
\end{aligned}$$

The motion of $\boldsymbol{\rho}_{st}$ has in the absence of interaction become unitary. It is the presence of the $\sum_{\alpha,\beta}$ that destroys the unitarity of (12.1). Equation (13) presents a curious generalization of the unitarity condition, to which it reduces in the absence of indices, and which begs to be brought within the compass of some kind of general theory. I expect to take up this challenge on another occasion.

Breuer & Petruccione write

$$\boldsymbol{\rho}_{st} = \sum_{\alpha,\beta} \mathsf{W}_{\alpha\beta}(t)\,\boldsymbol{\rho}_{s0}\,\mathsf{W}^+_{\alpha\beta}(t) \equiv \mathcal{V}(t)\boldsymbol{\rho}_{s0} \tag{14}$$

where $\mathcal{V}(t)$ is an operator (what Breuer & Petruccione call a "super-operator") that achieves a certain linear reorganization of the elements of $\boldsymbol{\rho}_{s0}$. Suppose, for example, that $\mathcal{H}_s$ is 2-dimensional, and that $\boldsymbol{\rho}_{s0}$ can be represented

$$\rho\!\!\!/_{s0} = \begin{pmatrix} \rho_{0,11} & \rho_{0,12} \\ \rho_{0,21} & \rho_{0,22} \end{pmatrix}$$

Then the upshot of (14) can be described

$$\begin{pmatrix} \rho_{t,11} \\ \rho_{t,12} \\ \rho_{t,21} \\ \rho_{t,22} \end{pmatrix} = \begin{pmatrix} V_{11,11}(t) & V_{11,12}(t) & V_{11,21}(t) & V_{11,22}(t) \\ V_{12,11}(t) & V_{12,12}(t) & V_{12,21}(t) & V_{12,22}(t) \\ V_{21,11}(t) & V_{21,12}(t) & V_{21,21}(t) & V_{21,22}(t) \\ V_{22,11}(t) & V_{22,12}(t) & V_{22,21}(t) & V_{22,22}(t) \end{pmatrix} \begin{pmatrix} \rho_{0,11} \\ \rho_{0,12} \\ \rho_{0,21} \\ \rho_{0,22} \end{pmatrix}$$

and abbreviated

$$\vec{\rho\!\!\!/}_{st} = \mathbb{V}(t)\,\vec{\rho\!\!\!/}_{s0}$$

Proceeding on the assumption that it is possible to write $\mathbb{V}(t) = \exp\big\{-i\mathbb{L}\,t\big\}$ we are led to a differential equation of "Markovian" form[10]

$$i\tfrac{d}{dt}\vec{\rho\!\!\!/}_{st} = \mathbb{L}\,\vec{\rho\!\!\!/}_{st}$$

This equation illustrates the explicit meaning of the equation that Breuer & Petruccione write

$$i\tfrac{d}{dt}\boldsymbol{\rho}_{st} = \mathcal{L}\,\boldsymbol{\rho}_{st} \tag{15}$$

and call the "Markovian quantum master equation." We undertake now to develop the explicit meaning of the $\mathcal{L}\,\boldsymbol{\rho}_{st}$.

Suppose $\mathcal{H}_s$ to be n-dimensional. Let operators $\mathsf{F}_i, i = 1, 2, \ldots, n^2$ span the "Liouville space" of linear operators on $\mathcal{H}_s$. Assume without loss of generality that the F_i are orthonormal in the tracewise sense

$$(\,\mathsf{F}_i, \mathsf{F}_j) \equiv \tfrac{1}{n}\,\mathrm{tr}\big\{\,\mathsf{F}_i^+\,\mathsf{F}_j\big\} = \delta_{ij}$$

Assume more particularly that $\mathsf{F}_1 = \mathsf{I}$. The remaining basis operators are then

[10] The i-factor is omitted in the literature. I have introduced it in recognition that we are, after all, doing quantum mechanics.

necessarily traceless: $(\mathsf{F}_1, \mathsf{F}_{i\neq1}) = 0 \sim \mathrm{tr}\big\{\mathsf{F}_{i\neq1}\big\}$.[11] For any linear operator A on $\mathcal{H}_s$ we have

$$\mathsf{A} = \sum_{i=1}^{n^2} \mathsf{F}_i\,(\mathsf{F}_i, \mathsf{A})$$

In particular, we have

$$\mathsf{W}_{\alpha\beta} = \sum_{i=1}^{n^2} \mathsf{F}_i\,(\mathsf{F}_i, \mathsf{W}_{\alpha\beta})$$

which by (14) gives

$$\begin{aligned}\mathcal{V}(t)\boldsymbol{\rho}_{s0} &= \sum_{\alpha,\beta} \mathsf{W}_{\alpha\beta}(t)\,\boldsymbol{\rho}_{s0}\,\mathsf{W}^+_{\alpha\beta}(t)\\ &= \sum_{i,j=1}^{n^2} \underbrace{\sum_{\alpha,\beta}(\mathsf{F}_i, \mathsf{W}_{\alpha\beta})(\mathsf{F}_j, \mathsf{W}_{\alpha\beta})^*}_{c_{ij}(t)}\;\mathsf{F}_i\,\boldsymbol{\rho}_{s0}\,\mathsf{F}^+_j\end{aligned}$$

in which notation (13) becomes

$$\sum_{\alpha,\beta} \mathsf{W}_{\alpha\beta}(t)\mathsf{W}^+_{\alpha\beta}(t) = \sum_{i,j=1}^{n^2} c_{ij}(t)\,\mathsf{F}_i\,\mathsf{F}^+_j = \mathsf{I}_s = \mathsf{F}_1$$

Taking now into account the unique simplicity of $\mathsf{F}_1 = \mathsf{I}$, we have

$$\begin{aligned}\tfrac{\partial}{\partial t}\,\boldsymbol{\rho}_{st} &= \dot{c}_{11}\boldsymbol{\rho}_{s0} + \sum_{i=2}^{n^2}\Big(\dot{c}_{i1}\,\mathsf{F}_i\,\boldsymbol{\rho}_{s0} + \dot{c}_{1i}\,\boldsymbol{\rho}_{s0}\,\mathsf{F}^+_j\Big) + \sum_{i,j=2}^{n^2}\dot{c}_{ij}\,\mathsf{F}_i\,\boldsymbol{\rho}_{s0}\,\mathsf{F}^+_j\\ &= \mathcal{L}\,\boldsymbol{\rho}_{st}\end{aligned}$$

---

[11] In the case $n = 2$ we might, for example, set (in matrix representation)

$$\mathbb{F}_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\mathbb{F}_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \mathbb{F}_3 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \mathbb{F}_4 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

These familiar matrices happen to be hermitian, but serve nevertheless to span the space of *all* 2×2 matrices; one has, for example,

$$\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} = \tfrac{1}{2}a\mathbb{F}_2 + i\tfrac{1}{2}a\mathbb{F}_3$$

A simple argument shows that one can *always* (for all n) impose a hermiticity requirement upon such $\mathbb{F}$ matrices.

All dynamical information is conveyed now by the functions $\dot{c}_{ij}(t)$. At $t = 0$ those become constants $a_{ij} \equiv \dot{c}_{ij}(0)$ and the preceding equation reads

$$\begin{aligned}\mathcal{L}\,\boldsymbol{\rho}_{s0} &= a_{11}\boldsymbol{\rho}_{s0} + \sum_{i=2}^{n^2}\Big(a_{i1}\,\mathsf{F}_i\,\boldsymbol{\rho}_{s0} + a_{1i}\,\boldsymbol{\rho}_{s0}\,\mathsf{F}_j^+\Big) + \sum_{i,j=2}^{n^2} a_{ij}\,\mathsf{F}_i\,\boldsymbol{\rho}_{s0}\,\mathsf{F}_j^+ \\ &= a_{11}\boldsymbol{\rho}_{s0} + \big(\mathsf{F}\,\boldsymbol{\rho}_{s0} + \boldsymbol{\rho}_{s0}\,\mathsf{F}^+\big) + \sum_{i,j=2}^{n^2} a_{ij}\,\mathsf{F}_i\,\boldsymbol{\rho}_{s0}\,\mathsf{F}_j^+ \end{aligned} \tag{16}$$

which serves to define the action of the "super-generator" $\mathcal{L}$. Here

$$\begin{aligned}\mathsf{F} &\equiv \sum_{i=2}^{n^2} a_{i1}\,\mathsf{F}_i \\ &\Downarrow \\ \mathsf{F}^+ &= \sum_{i=2}^{n^2} a_{1i}\,\mathsf{F}_i^+ \quad \text{since the hermiticity of } \|c_{ij}(t)\| \Rightarrow (a_{i1})^* = a_{1i}\end{aligned}$$

Resolving F into its hermitian and anti-hermitian parts

$$\mathsf{F} = \tfrac{1}{2}\big(\mathsf{F} + \mathsf{F}^+\big) + \tfrac{1}{2}\big(\mathsf{F} - \mathsf{F}^+\big) \equiv \mathsf{g} - i\,\mathsf{H}$$

we have

$$\begin{aligned}a_{11}\boldsymbol{\rho}_{s0} + \big(\mathsf{F}\,\boldsymbol{\rho}_{s0} + \boldsymbol{\rho}_{s0}\,\mathsf{F}^+\big) &= -i\big(\mathsf{H}\,\boldsymbol{\rho}_{s0} - \boldsymbol{\rho}_{s0}\,\mathsf{H}\big) + a_{11}\boldsymbol{\rho}_{s0} + \big(\mathsf{g}\,\boldsymbol{\rho}_{s0} + \boldsymbol{\rho}_{s0}\,\mathsf{g}\big) \\ &= -[\mathsf{H}, \boldsymbol{\rho}_{s0}] + \{\mathsf{G}, \boldsymbol{\rho}_{s0}\}\end{aligned}$$

with $$\mathsf{G} \equiv \tfrac{1}{2}a_{11}\,\mathsf{I} + \mathsf{g}$$
and find that (16) can be written

$$\mathcal{L}\,\boldsymbol{\rho}_{s0} = -i[\mathsf{H}, \boldsymbol{\rho}_{s0}] + \{\mathsf{G}, \boldsymbol{\rho}_{s0}\} + \sum_{i,j=2}^{n^2} a_{ij}\,\mathsf{F}_i\,\boldsymbol{\rho}_{s0}\,\mathsf{F}_j^+ \tag{17}$$

From
$$\boldsymbol{\rho}_{st} = e^{\mathcal{L}t}\boldsymbol{\rho}_{s0} = \boldsymbol{\rho}_{s0} + t\cdot\mathcal{L}\,\boldsymbol{\rho}_{s0} + \cdots$$
and the previously established fact that $\mathrm{tr}\big\{\boldsymbol{\rho}_{st}\big\} = \mathrm{tr}\big\{\boldsymbol{\rho}_{s0}\big\}$ we conclude that $\mathrm{tr}\big\{\mathcal{L}\,\boldsymbol{\rho}_{s0}\big\} = 0$, which by (17) entails

$$0 = \mathrm{tr}\bigg\{\Big(2\mathsf{G} + \sum_{i,j=2}^{n^2} a_{ij}\,\mathsf{F}_j^+\,\mathsf{F}_i\Big)\boldsymbol{\rho}_{s0}\bigg\} \quad : \quad \text{all } \boldsymbol{\rho}_{s0}$$

whence
$$\mathsf{G} = -\tfrac{1}{2}\sum_{i,j=2}^{n^2} a_{ij}\,\mathsf{F}_j^+\,\mathsf{F}_i$$

Equation (12) now presents what Breuer & Petruccione call the "first standard form"

$$\mathcal{L}\,\boldsymbol{\rho}_s = -i[\mathsf{H},\boldsymbol{\rho}_s] + \sum_{i,j=2}^{n^2} a_{ij}\Big(\mathsf{F}_i\,\boldsymbol{\rho}_s\,\mathsf{F}_j^+ - \tfrac{1}{2}\{\mathsf{F}_j^+\mathsf{F}_i,\boldsymbol{\rho}_s\}\Big) \tag{18}$$

of the description of the action achieved by the super-generator $\mathcal{L}$.

Further progress requires that we sharpen what we know about the coefficients

$$c_{ij}(t) \equiv \sum_{\alpha,\beta}(\mathsf{F}_i,\mathsf{W}_{\alpha\beta})(\mathsf{F}_j,\mathsf{W}_{\alpha\beta})^*$$

It is, as previously remarked, immediate that $\|c_{ij}(t)\|$ is hermitian. Moreover

$$\sum_{i,j} v_i^* c_{ij} v_j = \sum_{\alpha,\beta}\Big|\Big(\sum_k (v_k\mathsf{F}_k,\mathsf{W}_{\alpha\beta})\Big)\Big|^2 \geqslant 0 \quad : \quad \text{all complex vectors } v$$

so the eigenvalues of $\|c_{ij}(t)\|$ must all be non-negative. The same can, of course, be said of $\|a_{ij}\|$. And of the sub-matrix that results from restricting the range of i and j, as is done in (18).

Let u_{ip} be elements of the unitary matrix that diagonalizes $\|a_{ij}\|$:

$$a_{ij} = \sum_{p,q} u_{ip}\Lambda_{pq}\bar{u}_{jq} \quad \text{with} \quad \|\Lambda_{pq}\| = \begin{pmatrix} \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{n^2} \end{pmatrix}$$

Equation (18) becomes

$$\begin{aligned}
\mathcal{L}\,\boldsymbol{\rho}_s &= -i[\mathsf{H},\boldsymbol{\rho}_s] + \sum_{i,j,p,q=2}^{n^2} u_{ip}\Lambda_{pq}\bar{u}_{jq}\Big(\mathsf{F}_i\,\boldsymbol{\rho}_s\,\mathsf{F}_j^+ - \tfrac{1}{2}\{\mathsf{F}_j^+\mathsf{F}_i,\boldsymbol{\rho}_s\}\Big)\\
&= -i[\mathsf{H},\boldsymbol{\rho}_s] + \sum_{i,j,p,q=2}^{n^2} \lambda_p\delta_{pq}\Big(u_{ip}\mathsf{F}_i\,\boldsymbol{\rho}_s\bar{u}_{jq}\mathsf{F}_j^+ - \tfrac{1}{2}\{\bar{u}_{jq}\mathsf{F}_j^+u_{ip}\mathsf{F}_i,\boldsymbol{\rho}_s\}\Big)\\
&= -i[\mathsf{H},\boldsymbol{\rho}_s] + \sum_{p=2}^{n^2}\lambda_p\Big(\mathsf{A}_p\,\boldsymbol{\rho}_s\,\mathsf{A}_p^+ - \tfrac{1}{2}\mathsf{A}_p^+\mathsf{A}_p\,\boldsymbol{\rho}_s - \tfrac{1}{2}\boldsymbol{\rho}_s\,\mathsf{A}_p^+\mathsf{A}_p\Big)
\end{aligned} \tag{19}$$

with

$$\mathsf{A}_p = \sum_{i=2}^{n^2} u_{ip}\,\mathsf{F}_i$$

The operators $\mathsf{A}_p, p = 2,3,\dots,n^2$ are called "Lindblad" operators, and

$$\tfrac{d}{dt}\boldsymbol{\rho}_{st} = -i[\mathsf{H},\boldsymbol{\rho}_{st}] + \underbrace{\sum_{p=2}^{n^2}\lambda_p\Big(\mathsf{A}_p\,\boldsymbol{\rho}_{st}\,\mathsf{A}_p^+ - \tfrac{1}{2}\mathsf{A}_p^+\mathsf{A}_p\,\boldsymbol{\rho}_{st} - \tfrac{1}{2}\boldsymbol{\rho}_{st}\,\mathsf{A}_p^+\mathsf{A}_p\Big)}_{\mathcal{D}(\boldsymbol{\rho}_{st})} \tag{20}$$

is called the "Lindblad equation."[12] The H term generates a unitary motion which is distinct from that which in the absence of system-environmental interaction would have been generatd by H_s; the interaction term H_i was seen above to enter into the construction of $\mathsf{H} = i\frac{1}{2}\big(\mathsf{F}-\mathsf{F}^+\big)$. It is the "dissipator" $\mathcal{D}(\boldsymbol{\rho}_{st})$ that accounts for the non-unitarity of $\boldsymbol{\rho}_{s0}\longmapsto\boldsymbol{\rho}_{st}$.

Much freedom attended our selection of an orthonormal basis

$$\big\{\mathsf{I},\mathsf{F}_2,\mathsf{F}_3,\ldots\mathsf{F}_{n^2}\big\}$$

in the "Liouville space" of linear operators on $\mathcal{H}_s$, so the expression on the right side of (20) is highly non-unique. If, for example, we write

$$\sqrt{\lambda_p}\,\mathsf{A}_p=\sum_q u_{pq}\sqrt{\mu_q}\,\mathsf{B}_q$$

we obtain

$$\mathcal{D}(\boldsymbol{\rho}_{st})=\sum_{p=2}^{n^2}\lambda_p\Big(\mathsf{A}_p\,\boldsymbol{\rho}_{st}\,\mathsf{A}_p^+-\tfrac{1}{2}\mathsf{A}_p^+\mathsf{A}_p\,\boldsymbol{\rho}_{st}-\tfrac{1}{2}\boldsymbol{\rho}_{st}\,\mathsf{A}_p^+\mathsf{A}_p\Big)\tag{21}$$

$$\begin{aligned}=\sum_{p,q,r=2}^{n^2}&\Big(u_{pq}\sqrt{\mu_q}\,\mathsf{B}_q\boldsymbol{\rho}_{st}\bar{u}_{pr}\sqrt{\mu_r}\,\mathsf{B}_r^+\\&-\tfrac{1}{2}\bar{u}_{pr}\sqrt{\mu_r}\,\mathsf{B}_r^+\,u_{pq}\sqrt{\mu_q}\,\mathsf{B}_q\,\boldsymbol{\rho}_{st}-\tfrac{1}{2}\boldsymbol{\rho}_{st}\,\bar{u}_{pr}\sqrt{\mu_r}\,\mathsf{B}_r^+\,u_{pq}\sqrt{\mu_q}\,\mathsf{B}_q\Big)\end{aligned}$$

which, if we impose the unitarity assumption $\sum_p u_{pq}\bar{u}_{pr}=\delta_{qr}$, becomes

$$\mathcal{D}(\boldsymbol{\rho}_{st})=\sum_{q=2}^{n^2}\mu_q\Big(\mathsf{B}_q\,\boldsymbol{\rho}_{st}\,\mathsf{B}_q^+-\tfrac{1}{2}\mathsf{B}_q^+\mathsf{B}_q\,\boldsymbol{\rho}_{st}-\tfrac{1}{2}\boldsymbol{\rho}_{st}\,\mathsf{B}_q^+\mathsf{B}_q\Big)$$

which is structurally identical to (21). Or consider

$$\mathsf{A}_p\longmapsto\mathsf{A}_p+a_p\,\mathsf{I}$$

which (working again from (21)) sends

$$\begin{aligned}\mathcal{D}(\boldsymbol{\rho}_{st})\longmapsto\mathcal{D}(\boldsymbol{\rho}_{st})+\sum_{p=2}^{n^2}\lambda_p\Big\{&\Big(a_p\,\boldsymbol{\rho}_{st}\,\mathsf{A}_p^+-\tfrac{1}{2}\mathsf{A}_p^+a_p\,\boldsymbol{\rho}_{st}-\tfrac{1}{2}\boldsymbol{\rho}_{st}\,\mathsf{A}_p^+a_p\Big)\\&+\Big(\mathsf{A}_p\,\boldsymbol{\rho}_{st}\bar{a}_p-\tfrac{1}{2}\bar{a}_p\mathsf{A}_p\,\boldsymbol{\rho}_{st}-\tfrac{1}{2}\boldsymbol{\rho}_{st}\bar{a}_p\mathsf{A}_p\Big)\Big\}+\mathbf{0}\end{aligned}$$

$$\downarrow$$

$$\mathcal{D}(\boldsymbol{\rho}_{st})+\tfrac{1}{2}\sum_{p=2}^{n^2}\lambda_p\Big\{a_p\,[\boldsymbol{\rho}_{st},\mathsf{A}_p^+]-\bar{a}_p[\boldsymbol{\rho}_{st},\mathsf{A}_p]\Big\}$$

$$\downarrow$$

$$\mathcal{D}(\boldsymbol{\rho}_{st})+\Big[\,\boldsymbol{\rho}_{st},\tfrac{1}{2}\sum_{p=2}^{n^2}\lambda_p\Big(a_p\,\mathsf{A}_p^+-\bar{a}_p\mathsf{A}_p\Big)\Big]$$

[12] Breuer & Petruccione cite G. Lindblad, "On the generator of quantum mechanical semigroups," Commun. Math. Physics **48**, 119–130 (1976).

The additive term can be absorbed into a redefinition of the effective Hamiltonian:

$$\mathsf{H} \longrightarrow \mathsf{H}_{\text{adjusted}} \equiv \mathsf{H} + i\tfrac{1}{2}\sum_{p=2}^{n^2}\lambda_p\Big(a_p\,\mathsf{A}_p^+ - \bar{a}_p\,\mathsf{A}_p\Big)$$

NOTATIONAL REMARK: Breuer & Petruccione's notation $\mathcal{D}(\boldsymbol{\rho}_s)$ serves well enough to signify a "function of (the matrix elements of) an operator, though $\boldsymbol{\mathfrak{D}}(\boldsymbol{\rho}_s)$ would better emphasize that we are talking about an *operator valued* function of an operator. One would expect in that same spirit to write $\mathcal{V}(\boldsymbol{\rho}_s, t)$ and $\mathcal{L}(\boldsymbol{\rho}_s)$ where Breuer & Petruccione elect to write $\mathcal{V}(t)\boldsymbol{\rho}_s$ and $\mathcal{L}\boldsymbol{\rho}_s$, even though the "super-operators" $\mathcal{V}(t)$ and $\mathcal{L}$ are defined always by their functional action, never as stand-alone objects. It's my guess that they do so to motivate the train of thought that follows from writing $\mathcal{V}(t) = \exp\big\{\mathcal{V}\,t\big\}$. In these respects the matrix notation to which I alluded on page 16 provides a more frankly informative account of the situation. It should be noted also that, while $\mathcal{V}(t)$, $\mathcal{L}$ and $\mathcal{D}$ act upon the instantaneous value of $\boldsymbol{\rho}_s$, each acquires its specific structure from spectral properties of $\boldsymbol{\rho}_{m0}$ and of $\mathsf{H}_{\text{composite}}$.