



# Motion of a charged particle in the static fields of an infinite straight wire

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The nonrelativistic trajectory of a point charge  $q$  in the magnetic field of a steady line current  $I$  is characterized by the three components of its initial velocity. The motion is periodic in the cylindrical coordinates  $s$ ,  $\phi$ , and  $z$ , describing, in the generic case, a kind of “double helix,” with one helix serving as a guide while the other winds around it. A positive charge “drifts” in the direction of the current (a negative charge goes the other way). The inclusion of a uniform line charge  $\lambda$  (coinciding with the current) does not alter the motion qualitatively, but it does change the drift velocity, and can even reduce it to zero, collapsing the trajectory to the surface of a toroid.

The relativistic treatment modifies and illuminates these results. © 2022 Published under an exclusive license by American Association of Physics Teachers.

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## I. INTRODUCTION

In this paper, we study the motion of a charged particle in the magnetic field of a long straight steady current. The motion is (perhaps surprisingly) periodic in the radial cylindrical coordinate and also in the longitudinal and azimuthal velocities.

In Sec. II, we obtain the nonrelativistic equations of motion and their first integrals. In Sec. III, we examine three special cases and adopt useful conventions for numerical work. In Sec. IV, we endow the wire with a uniform static charge (in addition to its current); this introduces an electric force, but the main effect is to alter the drift velocity. In Sec. V, we present a relativistic treatment, including both electric and magnetic fields. Some of these results have appeared in the recent literature, as indicated in the endnotes, but we have endeavored to make our discussion self-contained.

## II. EQUATIONS OF MOTION

The magnetic field of an infinite straight wire carrying a steady current  $I$ , in cylindrical coordinates  $(s, \phi, z)$ , with  $I$  along the  $z$  axis, is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}. \quad (1)$$

The force on a charge  $q$  moving with velocity

$$\mathbf{v} = \dot{s} \hat{s} + s\dot{\phi} \hat{\phi} + \dot{z} \hat{z} \quad (2)$$

is

$$\begin{aligned} \mathbf{F} &= q(\mathbf{v} \times \mathbf{B}) = \frac{\mu_0 q I}{2\pi s} (\dot{s} \hat{s} + s\dot{\phi} \hat{\phi} + \dot{z} \hat{z}) \times \hat{\phi} \\ &= \frac{\mu_0 q I}{2\pi s} (\dot{s} \hat{z} - \dot{z} \hat{s}) = m\mathbf{a}. \end{aligned} \quad (3)$$

The acceleration (in cylindrical coordinates) is

$$\mathbf{a} = (\ddot{s} - s\dot{\phi}^2) \hat{s} + (s\ddot{\phi} + 2\dot{s}\dot{\phi}) \hat{\phi} + \ddot{z} \hat{z}, \quad (4)$$

so the equations of motion are

$$[1] \ddot{s} - s\dot{\phi}^2 = -\alpha \frac{\dot{z}}{s}, \quad [2] s\ddot{\phi} + 2\dot{s}\dot{\phi} = 0, \quad [3] \ddot{z} = \alpha \frac{\dot{s}}{s}, \quad (5)$$

where

$$\alpha \equiv \frac{\mu_0 q I}{2\pi m} \quad (6)$$

is a constant characterizing the strength of the force.

*Question:* What is the trajectory of the particle?<sup>1-6</sup>

Multiplying Eq. (5-[2]) by  $s$ , we get

$$s^2 \ddot{\phi} + 2s\dot{s}\dot{\phi} = \frac{d}{dt}(s^2 \dot{\phi}) = 0 \quad \Rightarrow \quad s^2 \dot{\phi} = \frac{\ell}{m} \quad (7)$$

is a constant of the motion (physically,  $\ell$  is the  $z$  component of the angular momentum). Meanwhile, Eq. (5-[3]) indicates

$$\frac{d}{dt}(\dot{z}) = \frac{d}{dt}(\alpha \ln s) \quad \Rightarrow \quad \dot{z} = \alpha \ln(s/s_0), \quad (8)$$

where  $s_0$  is a second constant of the motion—not to be confused with  $s(0)$  (physically,  $s_0$  is related to the momentum conjugate to  $z$ ). Instead of using Eq. (5-[1]) to get  $\dot{s}$ , it is a little quicker to exploit the fact that magnetic forces do no work, so the kinetic energy is conserved:

$$E = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2), \quad (9)$$

or

$$E = \frac{1}{2} m \dot{s}^2 + \frac{\ell^2}{2ms^2} + \frac{1}{2} m \alpha^2 [\ln(s/s_0)]^2. \quad (10)$$

Serendipitously, this is nothing but the equation of motion for a mass  $m$  in the one-dimensional potential

$$V(s) = \frac{\ell^2}{2ms^2} + \frac{1}{2} m \alpha^2 [\ln(s/s_0)]^2. \quad (11)$$

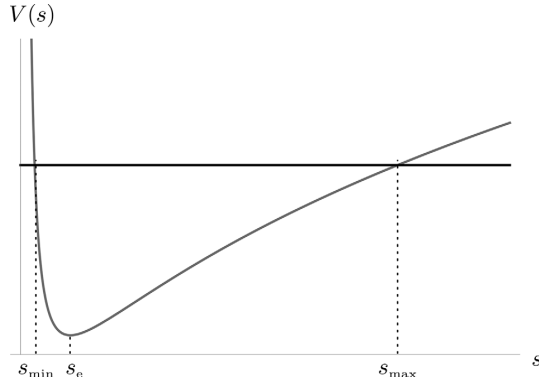


Fig. 1. The potential in Eq. (11).

This potential is plotted in Fig. 1. The equilibrium radius  $s_e$  is given by  $dV/ds = 0$ :

$$\left(\frac{\ell}{ms_e}\right)^2 = \ln\left(\frac{s_e}{s_0}\right). \quad (12)$$

Provided that

$$E > V(s_e) = \frac{\ell^2}{2ms_e^2} \left[ 1 + \left(\frac{\ell}{ms_e}\right)^2 \right], \quad (13)$$

the particle slides back and forth, periodically, between the turning points  $s_{\min}$  and  $s_{\max}$ . But if  $s(t)$  is periodic, so too are  $\dot{\phi}$  (Eq. (7)) and  $\dot{z}$  (Eq. (8)). This means that  $\phi(t)$  and  $z(t)$  both take the “quasiperiodic” form<sup>7</sup>

$$\kappa t + f(t), \quad (14)$$

where  $\kappa$  is a constant and  $f(t)$  is periodic ( $f(t+T) = f(t)$ , with  $T$  the period of  $s(t)$ ). Except for a constant azimuthal velocity and a constant longitudinal drift, all three coordinates oscillate (at the same frequency). The entire motion is a kind of wobbling deformed helix, or “helicoid.”<sup>3</sup>

### III. SPECIAL CASES AND NUMERICAL SOLUTIONS

#### A. The Helix

If the amplitude of the radial oscillations is zero, then it is a *true* helix ( $\dot{\phi}$  and  $\dot{z}$  are constants):<sup>8</sup>

$$s(t) = s_e, \quad \dot{\phi}(t) = \frac{\ell}{ms_e}, \quad \dot{z}(t) = \frac{\ell^2}{\alpha m^2 s_e^2}, \quad (15)$$

(see Fig. 2).

#### B. Small amplitudes

The second derivative at the equilibrium point (the “spring constant” for small oscillations) is

$$k \equiv \left. \frac{d^2V}{ds^2} \right|_{s_e} = \frac{m\alpha^2}{s_e^2} + \frac{2\ell^2}{ms_e^4}, \quad (16)$$

so the (angular) frequency for small amplitudes is

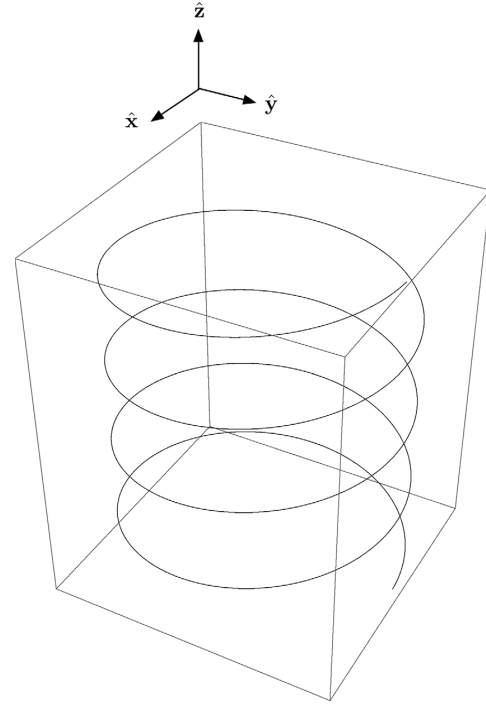


Fig. 2. The helix, case A.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\alpha^2}{s_e^2} + \frac{2\ell^2}{m^2 s_e^4}} = \frac{\alpha}{s_e} \sqrt{1 + 2 \ln(s_e/s_0)}. \quad (17)$$

The trajectory is a kind of double helix, with a small coil winding around the larger helix of Sec. III A (see Fig. 3).

#### C. Zero angular momentum

If  $\ell = 0$ ,  $\dot{\phi} = 0$ , and the motion is confined to a plane containing the  $z$  axis, the equilibrium radius is  $s_e = s_0$  (Eq. (12)), and the frequency for small oscillations is  $\omega = \alpha/s_0$  (Eq. (17)).<sup>6</sup> The trajectories are cycloid-like (see Fig. 4).

#### D. Numerical solutions

The general solution depends on the parameter  $\alpha$  (Eq. (6)), with the dimensions of speed. For numerical work, we might as well set<sup>9</sup>

$$\alpha \equiv 1; \quad (18)$$

this just defines a convenient unit for velocity. We will also set  $m = 1$ , establishing the unit of mass. Six initial conditions determine the trajectory:

(1)  $s(0)$  (the initial radial position); we can use this as the unit of length, so

$$s(0) \equiv 1 \quad (19)$$

(the unit of time, then, is  $s(0)/\alpha$ ).

(2)  $\phi(0)$  (the initial azimuthal angle); we might as well set this to zero

$$\phi(0) \equiv 0 \quad (20)$$

(this just establishes the orientation of the axes).

(3)  $z(0)$  (the initial value of  $z$ ); again, we might as well set this to zero

$$z(0) \equiv 0 \quad (21)$$

(that just picks the position of the origin along the  $z$  axis).

(4)  $\dot{s}(0)$  (the initial radial velocity):

$$\dot{s}(0) = \sqrt{2E - \ell^2 - (\ln s_0)^2}. \quad (22)$$

(5)  $\dot{\phi}(0)$  (the initial angular velocity):

$$\dot{\phi}(0) = \ell. \quad (23)$$

(6)  $\dot{z}(0)$  (the initial longitudinal velocity):

$$\dot{z}(0) = -\ln s_0. \quad (24)$$

Thus, each trajectory is characterized by three numbers, which we can take to be  $E$ ,  $\ell$ , and  $s_0$  (or, if you prefer,  $\dot{s}(0)$ ,  $\dot{\phi}(0)$ , and  $\dot{z}(0)$ ). In Figs. 2–4, we illustrate each of the trajectories discussed.<sup>10,11</sup>

### E. Period and drift velocity

For a given trajectory, the radius is localized ( $s_{\min} \leq s \leq s_{\max}$ ), but the  $z$  coordinate goes from  $-\infty$  to  $\infty$ . The period of the oscillations is

$$T = \int_0^T dt = 2 \int_{s_{\min}}^{s_{\max}} \frac{ds}{\dot{s}} = 2 \int_{s_{\min}}^{s_{\max}} \frac{ds}{\sqrt{(2E/m) - (\ell/ms)^2 - [\alpha \ln(s/s_0)]^2}}. \quad (25)$$

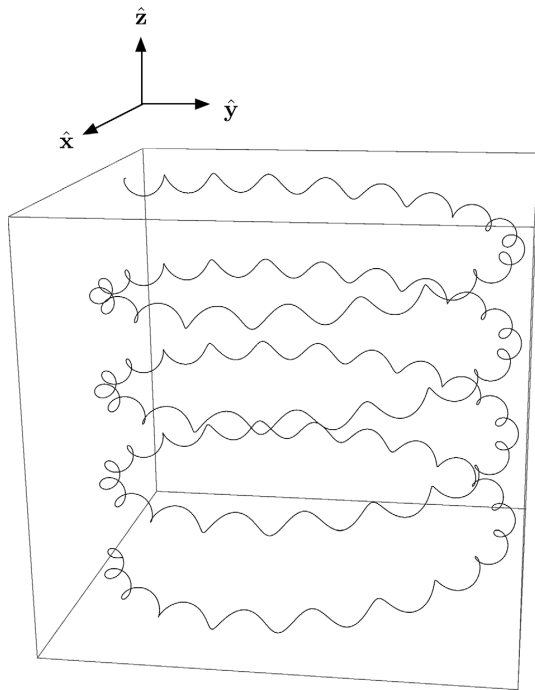


Fig. 3. The “double helix,” case B.

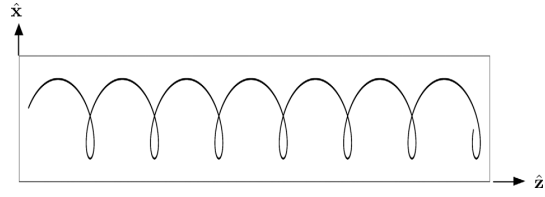


Fig. 4. The quasi-cycloid, case C.

The longitudinal drift velocity (the average value of  $\dot{z}$ ) is

$$v_d = \frac{\Delta z}{T}, \quad (26)$$

where

$$\Delta z = 2[z(s_{\max}) - z(s_{\min})] \quad (27)$$

is the change in  $z$  during one full cycle.<sup>12</sup>

$$v_d = \frac{2}{T} \int_{s_{\min}}^{s_{\max}} \frac{dz}{ds} ds = \frac{2}{T} \int_{s_{\min}}^{s_{\max}} \frac{\dot{z}}{\dot{s}} ds = \frac{2\alpha}{T} \int_{s_{\min}}^{s_{\max}} \frac{\ln(s/s_0)}{\sqrt{(2E/m) - (\ell/ms)^2 - [\alpha \ln(s/s_0)]^2}} ds. \quad (28)$$

The drift velocity is in the direction of the current if the charge is positive. (If it is negative,  $v_d$  is the opposite to  $I$ .) Here is the proof.<sup>5</sup>

**Proof:** Rewriting  $\Delta z$  (Eq. (27)), we get

$$\Delta z = \int_0^T \dot{z} dt. \quad (29)$$

From Eqs. (5) and (7),

$$\begin{aligned} \alpha \dot{z} &= -s(\ddot{s} - s\dot{\phi}^2) = s^2\dot{\phi}^2 - s\ddot{s} \\ &= \left(\frac{\ell}{ms}\right)^2 - \frac{d}{dt}(s\dot{s}) + \dot{s}^2, \end{aligned} \quad (30)$$

so

$$\Delta z = \frac{1}{\alpha} \int_0^T \left[ \dot{s}^2 + \left(\frac{\ell}{ms}\right)^2 \right] dt - \frac{1}{\alpha} s\dot{s} \Big|_0^T. \quad (31)$$

But the last term is zero, since  $s(t)$  is periodic, and the remaining integral is positive definite (except for the trivial case of a particle at rest or with no charge), so  $\Delta z$  shares the sign of  $\alpha$  (positive if  $ql > 0$ , minus if  $ql < 0$ ).

### IV. ADDING AN ELECTRIC FIELD

Suppose the wire carries a uniform line charge  $\lambda$  in addition to the steady current  $I$ . There is now a static electric field

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{s}, \quad (32)$$

as well as the magnetic field (Eq. (1)), and the net force is

$$\begin{aligned}
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \\
&= \frac{q\lambda}{2\pi\epsilon_0 s} \hat{\mathbf{s}} + \frac{\mu_0 q I}{2\pi s} (\dot{s} \hat{\mathbf{s}} + s \dot{\phi} \hat{\phi} + \dot{z} \hat{\mathbf{z}}) \times \hat{\phi}, \\
&= \frac{\mu_0 q I}{2\pi s} \left[ \dot{s} \hat{\mathbf{z}} + \left( \frac{\lambda c^2}{I} - \dot{z} \right) \hat{\mathbf{s}} \right] = m\mathbf{a}, \tag{33}
\end{aligned}$$

where  $c = 1/\sqrt{\epsilon_0 \mu_0}$  is the speed of light. The first equation of motion is now

$$[I'] \ddot{s} - s \dot{\phi}^2 = -\alpha \frac{(\dot{z} - \eta c)}{s}, \tag{34}$$

where

$$\eta \equiv \frac{\lambda c}{I}. \tag{35}$$

The other two are unchanged (Eq. (5)), and hence so too are the first integrals (7) and (8):

$$s^2 \dot{\phi} = \frac{\ell}{m} \quad \text{and} \quad \dot{z} = \alpha \ln(s/s_0). \tag{36}$$

However, for future convenience, we write the latter as

$$\dot{z} = \alpha \ln(s/\tilde{s}_0) + c\eta, \tag{37}$$

where  $\tilde{s}_0 \equiv s_0 e^{c\eta/\alpha}$ .

Conservation of energy now includes the electric potential energy:<sup>13</sup>

$$\begin{aligned}
E &= \frac{1}{2} m v^2 - \frac{q\lambda}{2\pi\epsilon_0} \ln(s/\tilde{s}_0) \\
&= \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - mc\alpha\eta \ln(s/\tilde{s}_0), \tag{38}
\end{aligned}$$

or, introducing a new energy  $\tilde{E}$  by subtracting off the irrelevant constant,

$$\begin{aligned}
\tilde{E} &\equiv E - \frac{1}{2} m c^2 \eta^2 \\
&= \frac{1}{2} m \dot{s}^2 + \frac{\ell^2}{2ms^2} + \frac{1}{2} m \alpha^2 [\ln(s/\tilde{s}_0)]^2. \tag{39}
\end{aligned}$$

The radial motion is that of a particle of energy  $\tilde{E}$  in a one-dimensional potential well

$$V(s) = \frac{\ell^2}{2ms^2} + \frac{1}{2} m \alpha^2 [\ln(s/\tilde{s}_0)]^2, \tag{40}$$

the same form as we had *without* the electric field (compare Eq. (11))!

This implies that there is a mapping between solutions to the original problem and solutions of the generalized system. Say that *without* the electric field, we have a solution

$$\{s_m(t), \phi_m(t), z_m(t)\}. \tag{41}$$

From this, we can construct a solution *with* the electric field

$$\{s_{em}(t), \phi_{em}(t), z_{em}(t)\} \tag{42}$$

(with  $s_0 \rightarrow \tilde{s}_0$  and  $E \rightarrow \tilde{E}$ ), such that

$$\begin{aligned}
s_{em}(t) &= s_m(t), \quad \phi_{em}(t) = \phi_m(t), \\
z_{em}(t) &= z_m(t) + c\eta t, \tag{43}
\end{aligned}$$

(Eq. (37)). For example, in case A (the helix), if the charge density is

$$\lambda = -\frac{2\pi\epsilon_0}{qms_e^2} \ell^2, \tag{44}$$

then (Eq. (15))

$$\dot{z}_{em} = \dot{z}_m + c\eta = 0, \tag{45}$$

and the helix collapses to a circle (Fig. 5); the magnetic force disappears entirely, and the orbital speed ( $v = s_e \dot{\phi}$ ) satisfies

$$-\frac{q\lambda}{2\pi\epsilon_0 s_e} = \frac{mv^2}{s_e}, \tag{46}$$

as it should (the electric attraction supplying the centripetal force). In the generic case, we can “stop the drift”—producing a localized “donut” trajectory (Fig. 6)—by subtracting  $v_d$  from  $v_z(0)$  and introducing an electric field (Eq. (32)) with

$$\frac{\lambda}{2\pi\epsilon_0} = -\frac{\alpha m}{q} v_d. \tag{47}$$

If all this comes as a surprise, remember that in relativity, a current in longitudinal motion acquires an electric charge.<sup>14</sup> Conversely, introducing  $\lambda$  amounts to adopting a moving reference frame. We will explore this further in Sec. V.

## V. RELATIVISTIC TREATMENT

Thus far, we have treated the motion nonrelativistically. In this section, we consider the fully relativistic case, and—for a change—we will use a Lagrangian approach. The relativistic Lagrangian for a point charge in electromagnetic fields specified by the potentials  $V$  and  $\mathbf{A}$  is<sup>15</sup>

$$L = -mc^2 \sqrt{1 - v^2/c^2} - q(V - \mathbf{v} \cdot \mathbf{A}). \tag{48}$$

In our case

$$V = -\frac{\lambda}{2\pi\epsilon_0} \ln(s/a), \quad \mathbf{A} = -\frac{\mu_0 I}{2\pi} \ln(s/a) \hat{\mathbf{z}}, \tag{49}$$

where  $a$  is an arbitrary reference point (which we might as well take to be the same for  $V$  and  $\mathbf{A}$ ). Thus,

$$\begin{aligned}
L &= -mc^2 \sqrt{1 - \frac{1}{c^2} (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2)} \\
&\quad + \frac{q\lambda}{2\pi\epsilon_0} \ln(s/a) - \frac{q\mu_0 I}{2\pi} \dot{z} \ln(s/a). \tag{50}
\end{aligned}$$

Since  $\phi$  and  $z$  are ignorable coordinates, the associated canonical momenta are constants of the motion:

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \gamma m s^2 \dot{\phi}, \tag{51}$$

$$p_z \equiv \frac{\partial L}{\partial \dot{z}} = \gamma m \dot{z} - \frac{q\mu_0 I}{2\pi} \ln(s/a), \tag{52}$$

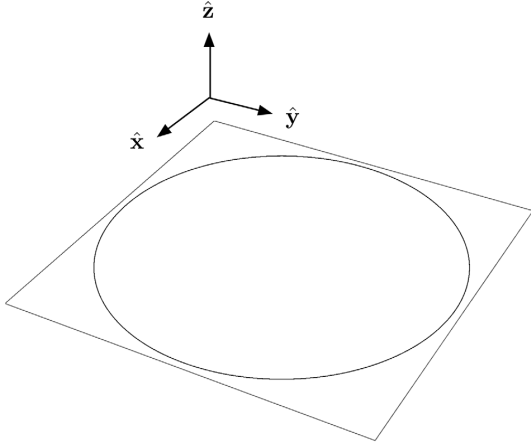


Fig. 5. The helix (Fig. 2) collapses to a circle.

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (53)$$

In addition, we have the conserved total energy<sup>16</sup>

$$E = \gamma mc^2 - \frac{q\lambda}{2\pi\epsilon_0} \ln(s/a). \quad (54)$$

From Eq. (54), we solve for  $\gamma$ :

$$\gamma = \frac{1}{mc^2} (E + \alpha \eta mc \ln(s/a)), \quad (55)$$

where (as before)

$$\alpha \equiv \frac{\mu_0 q l}{2\pi m}, \quad \eta \equiv \frac{\lambda c}{l}. \quad (56)$$

Expressing  $\dot{\phi}$  and  $\dot{z}$  in terms of  $s$  (and the conserved quantities):

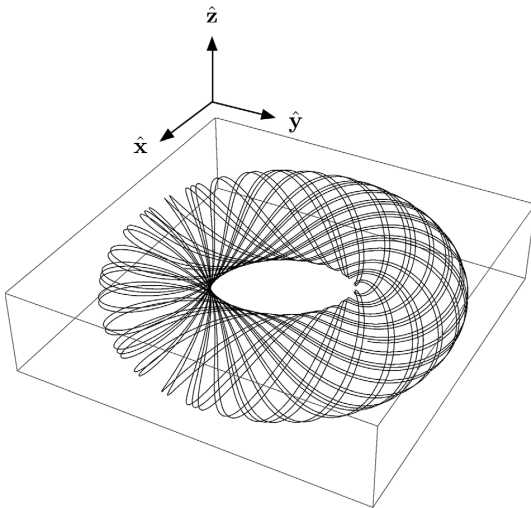


Fig. 6. The double helix shrinks to a donut.

$$\dot{\phi} = \frac{c^2 p_\phi}{s^2 (E + \alpha \eta mc \ln(s/a))}, \quad (57)$$

$$\dot{z} = \frac{c^2 (p_z + \alpha m \ln(s/a))}{(E + \alpha \eta mc \ln(s/a))}. \quad (58)$$

As for  $\dot{s}$ , Eq. (55) indicates

$$\begin{aligned} \frac{1}{\gamma^2} &= 1 - \frac{v^2}{c^2} = 1 - \frac{1}{c^2} (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) \\ &= \frac{m^2 c^4}{(E + \alpha \eta mc \ln(s/a))^2}, \end{aligned} \quad (59)$$

so substituting in Eqs. (57) and (58),

$$\dot{s}^2 = c^2 - c^4 \frac{(mc)^2 + (p_\phi/s)^2 + (p_z + \alpha m \ln(s/a))^2}{(E + \alpha \eta mc \ln(s/a))^2}. \quad (60)$$

This can be written in a more suggestive way

$$\frac{1}{2} mc^2 = \frac{1}{2} m \dot{s}^2 + V_{\text{eff}}(s), \quad (61)$$

where

$$V_{\text{eff}} \equiv \frac{1}{2} mc^4 \frac{[(mc)^2 + (p_\phi/s)^2 + (p_z + \alpha m \ln(s/a))^2]}{(E + \alpha \eta mc \ln(s/a))^2}. \quad (62)$$

Equation (61) can be read as a (*non-relativistic!*) one-dimensional problem in which a particle of total energy  $1/2 mc^2$  rocks back and forth in the potential well  $V_{\text{eff}}$ .

In particular, if  $\eta = 0$  (the purely magnetic case), then

$$\begin{aligned} V_{\text{eff}} &= \frac{mc^4}{2E^2} [(mc)^2 + (p_\phi/s)^2 + (p_z + \alpha m \ln(s/a))^2] \\ &= A + \frac{B}{s^2} + C [\ln(s/s_1)]^2, \end{aligned} \quad (63)$$

which (except for the irrelevant constant  $A$ ) has the same form as Eq. (11); again  $s(t)$  is periodic, and so too are  $\dot{\phi}$  and  $\dot{z}$ . Qualitatively, the trajectories are the same as in the nonrelativistic case. The situation is more complicated, when  $\eta$  is nonzero (i.e., with the electric field included), for now  $V_{\text{eff}}(s)$  does not go to infinity as  $s \rightarrow 0$  and  $s \rightarrow \infty$ . Instead, if  $p_\phi \neq 0$

$$V_{\text{eff}} \rightarrow \begin{cases} \infty, & \text{as } s \rightarrow 0, \\ \frac{1}{\eta^2} \left( \frac{1}{2} mc^2 \right), & \text{as } s \rightarrow \infty, \end{cases} \quad (64)$$

whereas if  $p_\phi = 0$

$$V_{\text{eff}} \rightarrow \frac{1}{\eta^2} \left( \frac{1}{2} mc^2 \right) \quad (65)$$

in both directions. Moreover,  $V_{\text{eff}}$  has an infinite spike at

$$s_{\text{sp}} = ae^{-E/\alpha \eta mc}, \quad (66)$$

where the denominator vanishes. In terms of  $s_{\text{sp}}$ ,

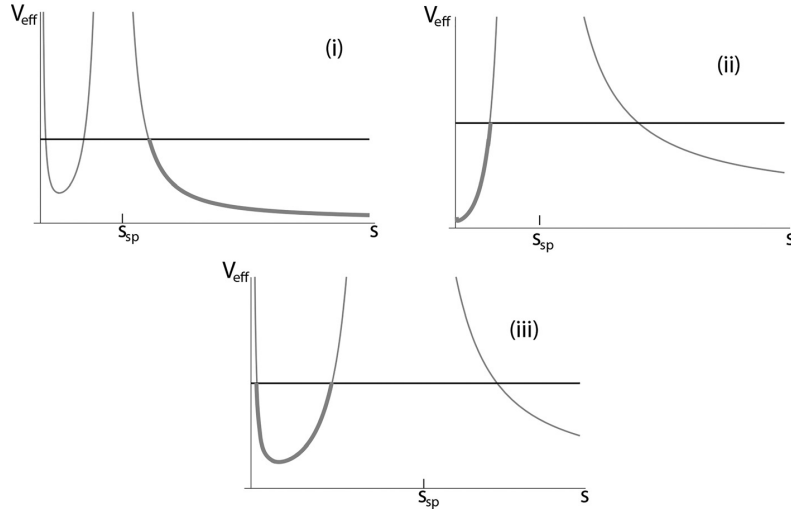


Fig. 7. Typical graphs of  $V_{\text{eff}}(s)$ , for (i)  $p_\phi \neq 0$  and  $\alpha\eta > 0$ ; (ii)  $p_\phi = 0$  and  $\alpha\eta < 0$ ; (iii)  $p_\phi \neq 0$  and  $\alpha\eta < 0$ . The horizontal line is at  $1/2mc^2$ . In the first graph, the inner region is inaccessible, the electric force is repulsive, and the charge flies off to infinity; in the second graph, the outer region is inaccessible, the electric force is attractive, and the charge falls to the center; in the third graph, outer region is inaccessible, the electric force is attractive, and the charge oscillates (in  $s$ ). The heavy line indicates the physical region.

$$\gamma = \frac{\alpha\eta}{c} \ln(s/s_{\text{sp}}), \quad (67)$$

and since for physically possible motion  $\gamma > 1$ , it follows that

$$\text{if } \alpha\eta > 0, \quad \text{then } s > s_{\text{sp}}, \quad (68)$$

$$\text{if } \alpha\eta < 0, \quad \text{then } s < s_{\text{sp}}. \quad (69)$$

In the first case  $\lambda q > 0$ , the electrical force is repulsive, and the motion is unbounded; in the second case  $\lambda q < 0$ , the electrical force is attractive, and the motion is bounded (in  $s$ ). Typical plots are shown in Fig. 7.

An illuminating way to think about these results is to imagine starting with a combination  $(\lambda, I)$ , and performing a Lorentz boost along the  $z$  direction. Because

$$J^\mu = (c\rho, \mathbf{J}) = \left( \frac{c\lambda}{A}, \frac{\mathbf{I}}{A} \right), \quad (70)$$

(where  $A$  is the cross-sectional area of the “wire”) constitutes a four-vector,  $(c\rho)^2 - J^2$ , and hence  $(c\lambda)^2 - I^2$  is Lorentz invariant. It could be time-like (or shall we call it “charge-like”?), or spacelike (“current-like”), and in the former case “forward” ( $\lambda > 0$ ) or “backward”:<sup>17</sup>

- (i) charge-like, forward  $(c\lambda)^2 - I^2 > 0, \lambda > 0$ ,
- (ii) charge-like, backward  $(c\lambda)^2 - I^2 > 0, \lambda < 0$ ,
- (iii) current-like,  $(c\lambda)^2 - I^2 < 0$ .

By suitable Lorentz boost a forward charge-like configuration can be transformed to any other forward charge-like configuration  $(\lambda', I')$  with  $(c\lambda')^2 - I'^2 = (c\lambda)^2 - I^2$ , a backward charge-like configuration can be transformed into any other backward charge-like configuration with  $(c\lambda')^2 - I'^2 = (c\lambda)^2 - I^2$ , and with a boost plus a rotation a current-like configuration can be transformed into any other current-like configuration with  $(c\lambda')^2 - I'^2 = (c\lambda)^2 - I^2$ . Therefore, there

are really just *three* distinct régimes here, and if we can find the trajectories for the “seed” configurations:

- (i)  $I = 0, \lambda q > 0$ ,
- (ii)  $I = 0, \lambda q < 0$ ,
- (iii)  $\lambda = 0, I \neq 0$ ,

we can construct the trajectory for *any*  $(\lambda, I)$  by Lorentz transformation. These are the régimes illustrated in Fig. 7: (i) runaway repulsion to  $s = \infty$ , (ii) attraction to the axis, and (iii) quasi-oscillatory motion with bounded  $s$ .

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## AUTHOR DECLARATIONS

### Conflict of Interest

There are no conflicts of interest to declare.

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<sup>1</sup>M. Müller and K. Dietrich, “Classical motion of a charged particle in the magnetic field of a rectilinear current,” *Z. Phys. D* **33**, 101–107 (1995). This paper allows the particle to carry a magnetic dipole moment in addition to its charge.

<sup>2</sup>D. Yafaev, “A particle in a magnetic field of an infinite rectilinear current,” *Math. Phys., Anal. Geom.* **6**, 219–230 (2003).

<sup>3</sup>J. Aguirre, A. Luque, and D. Peralta-Salas, “Motion of charged particles in magnetic fields created by symmetric configurations of wires,” *Physica D* **239**, 654–674 (2010).

<sup>4</sup>A. Prentice, M. Fatuzzo, and T. Toepker, “Charged particle dynamics in the magnetic field of a long straight current-carrying wire,” *Phys. Teach.* **53**, 34–37 (2015). This paper treats the two-dimensional example (case C). In an “Addendum” (*Phys. Teach.* **53**, 262 (2015)) the authors check that kinetic energy is conserved in this problem.

<sup>5</sup>H. Essén and A. B. Nordmark, “Drift velocity of charged particles in magnetic fields and its relation to the direction of the source current,” *Eur. Phys. J. D* **70**, 198–208 (2016).

<sup>6</sup>M. Asadi-Zeydabadi and C. S. Zaidins, “The trajectory of a charged particle in the magnetic field of an infinite current carrying wire in the nonrelativistic limit,” *Results Phys.* **12**, 2213–2217 (2019). This paper treats the two-dimensional example (Case C).

<sup>7</sup>This follows from integrating the Fourier expansion  $\dot{\phi}(t) = c_0 + \sum_{n=1}^{\infty} [c_n \cos(\frac{2\pi n}{T}t) + d_n \sin(\frac{2\pi n}{T}t)]$ .

<sup>8</sup>D. J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Cambridge U. P., Cambridge, 2017), Problem 5.39.

<sup>9</sup>What if  $\alpha$  is negative? Rotating the apparatus by  $180^\circ$  would reverse the sign of the current and render  $\alpha$  positive, while carrying one possible trajectory into another, so there is no loss of generality in assuming  $\alpha$  is positive.

<sup>10</sup>To draw the figures we used A. Chambliss and J. Franklin, “A magnetic velocity Verlet method,” *Am. J. Phys.* **88**, 1075–1082 (2020).

<sup>11</sup>See the supplementary material at <https://www.scitation.org/doi/suppl/10.1119/5.0077042> for the Mathematica code used to create the figures.

<sup>12</sup>In Case C ( $\ell = 0$ ) these integrals can be evaluated in closed form:  $T = \frac{2\pi\lambda v}{\alpha} I_0(v/\alpha)$ ,  $v_d = v \frac{I_1(v/\alpha)}{I_0(v/\alpha)}$ , where  $I_n$  is the modified Bessel function of order  $n$ , and  $v$  is the (constant) speed of the charge. These results were obtained by Essén and Nordmark (Ref. 5), but with unfortunate typos (their Eqs. (73) and (74)); they were derived in a different way by Asadi-Zeydabadi and Zaidins (Ref. 6).

<sup>13</sup>We might as well use  $\bar{s}_0$  as the zero for electric potential. You can use a different reference point if you like: the equations will be a little more cumbersome, but the result is the same.

<sup>14</sup>E. M. Purcell and D. J. Morin, *Electricity and Magnetism*, 3rd ed. (Cambridge U. P., Cambridge, 2013), Sec. 5.9.

<sup>15</sup>J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, New York, 1999), Eq. (12.12).

<sup>16</sup>Jackson (Ref. 15), Eq. (12.17).

<sup>17</sup>The “light-like” case ( $(c\lambda)^2 - l^2 = 0$ ) is qualitatively similar, with the exception of the trivial limit  $\lambda = 0$ ,  $l = 0$ , for which the trajectory is of course a straight line, with constant velocity.



**Canal Ray Tube**

This form of cathode ray tube was developed by Eugen Goldstein in 1886. The electrode in the middle serves as the cathode, and attracts positive ions to it. These ions may be produced by a cascade effect in which a few existing positive ions are accelerated toward the cathode. The central electrode is perforated with holes, and some of the positive ions pass through it. Once they are on the other side of the electrode they slow down by interacting with the residual gas ions and produce a faint flow in the process. This is typically blue, as the majority of the gas is nitrogen. This tube is in the Greenslade Collection. (Picture and text by Thomas B. Greenslade, Jr., Kenyon College)