RESOURCE LETTER

Resource Letter EM-1: Electromagnetic Momentum

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This Resource Letter surveys the literature on momentum in electromagnetic fields, including the general theory, the relation between electromagnetic momentum and vector potential, "hidden" momentum, the 4/3 problem for electromagnetic mass, and the Abraham–Minkowski controversy regarding the field momentum in polarizable and magnetizable media.

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I. INTRODUCTION

According to classical electrodynamics, electric and magnetic fields (**E** and **B**) store linear momentum, which must be included if the total momentum of a system is to be conserved. Specifically, the electromagnetic momentum per unit volume is

$$\mathbf{g} = \epsilon_0(\mathbf{E} \times \mathbf{B}), \tag{1}$$

as first proposed by Poynting (Refs. 30–32). Field momentum is most dramatically demonstrated in the laboratory by the pressure of light on an absorbing or reflecting surface. [In 1619 Kepler suggested that the pressure of light explains why comet tails point away from the sun (Ref. 29). The theory was developed by Maxwell (Ref. 10) and confirmed experimentally by Lebedew (Ref. 25) and Nichols and Hull (Ref. 28). Some introductory textbooks offer a quick qualitative explanation for the pressure exerted on a perfect conductor: **E** drives charge in (say) the x direction and **B** (in the y direction) then exerts a force in the z direction. This naive argument is faulty (Refs. 27, 33, and 16).]

But the notion that fields carry momentum leads to several intriguing problems, some of which are not entirely resolved after more than a century of debate.

(1) For a point charge q in an external field represented by the vector potential **A**, the electromagnetic momentum is $q\mathbf{A}$. This suggests that **A** can be thought of as "potential momentum per unit charge," just as the scalar potential V is "potential energy per unit charge." But this interpretation raises questions of its own, and it has never been universally accepted.

(2) According to Eq. (1), even purely *static* fields can store momentum. How can a system at rest carry momentum? It *cannot* … there must be some compensating *non*-electromagnetic momentum in such systems. But locating this "hidden momentum" can be subtle and difficult.

(3) A moving charge drags around the momentum in its fields, which means (in effect) that it has "extra" *mass*. But this "electromagnetic mass" is inconsistent with what you get from the *energy* in the fields (using Einstein's formula $E = mc^2$)—by a notorious factor of 4/3, in the case of a spherical shell. Which mass (if either) is "correct"?

(4) Inside matter, which is subject to polarization and magnetization, the effective field momentum is modified. Minkowski proposed

$$\mathbf{g}_\mathrm{M} = (\mathbf{D} \times \mathbf{B}); \tag{2}$$

Abraham advocated

$$\mathbf{g}_\mathrm{A} = \frac{1}{c^2}(\mathbf{E} \times \mathbf{H}). \tag{3}$$

For over a century a debate has raged: which expression is right? Or are they perhaps *both* right, and simply describe different things? How can the question be settled, theoretically and experimentally? Although many distinguished authors claim to have resolved the issue, the dispute continues to this day.

In Section II, I summarize the theory. I then survey each of the four controversies described qualitatively above. In the final section, I briefly consider electromagnetic momentum in quantum mechanics, where the photon makes the story in some respects more concrete and intuitive.

II. THEORY

A. Nonrelativistic

Electrodynamics (Refs. 2 and 6) is based on Maxwell's equations, which tell us how the sources (charge density ρ

and current density **J**) generate electric and magnetic fields (**E** and **B**):

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

and the Lorentz force law, which tells us the force exerted by the fields on a point charge q moving with velocity **v**:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{5}$$

The homogeneous Maxwell equations (the two that do not involve ρ or **J**) allow us to express the fields in terms of scalar and vector potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{6}$$

Electromagnetic fields store energy and momentum (and for that matter also angular momentum). The energy per unit volume in the fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right); \tag{7}$$

and the momentum density is

$$\mathbf{g} = \epsilon_0 (\mathbf{E} \times \mathbf{B}). \tag{8}$$

The fields also *transport* energy and momentum from one place to another. The energy *flux* (energy per unit time, per unit area) is given by the Poynting vector,

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \tag{9}$$

($\mathbf{S} \cdot d\mathbf{a}$ is the energy per unit time transported through a "window" of area $d\mathbf{a}$). The momentum flux is related to the Maxwell stress tensor:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) \tag{10}$$

(Specifically, the momentum per unit time transported through a window $d\mathbf{a}$ is $-\overleftrightarrow{T} \cdot d\mathbf{a}$). For example, the energy and momentum per unit time radiated (to infinity) by a nonrelativistic point charge q are

$$\frac{dE}{dt} = \frac{\mu_0 q^2}{6\pi c} a^2, \tag{11}$$

$$\frac{d\mathbf{p}}{dt} = \frac{\mu_0 q^2}{6\pi c^3} a^2 \mathbf{v}, \tag{12}$$

where **v** is the velocity of the charge and **a** is its acceleration. [The uniqueness of these expressions [Eqs. (7)–(10)] is open to some question (Ref. 23), but I shall take them as definitions.]

Several conservation laws follow from Maxwell's equations. Local conservation of charge is expressed by the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{13}$$

The corresponding statement for electromagnetic energy is

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -(\mathbf{E} \cdot \mathbf{J}). \tag{14}$$

This is Poynting's theorem; $\mathbf{E} \cdot \mathbf{J}$ is the power per unit volume delivered by the fields to the electric charges. Except in regions where $\mathbf{E} \cdot \mathbf{J} = 0$ (empty space, for example) the electromagnetic energy by itself is not conserved, because the fields do work on the charges. Similarly,

$$\frac{\partial \mathbf{g}}{\partial t} - \nabla \cdot \overleftrightarrow{T} = -(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}). \tag{15}$$

Here $(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B})$ is the force per unit volume exerted by the fields on the electric charges. Except (for example) in empty space, electromagnetic momentum by itself is not conserved. [Nor, therefore, is *mechanical* momentum separately conserved. This means that Newton's Third Law (although it holds in electro*statics* and magneto*statics*) is *not* obeyed in electro*dynamics* (Refs. 22 and 2).]

As we shall see, it is no accident that the *same* quantity $(\mathbf{E} \times \mathbf{B})$ appears in the Poynting vector and in the momentum density (Ref. 1),

$$\mathbf{S} = c^2 \mathbf{g} \tag{16}$$

(or that the same quantity $\overleftrightarrow{T}$ plays a dual role as force-per-unit-area and momentum flux).

B. Relativistic

1. Notation

The theory is more elegant in covariant (relativistic) notation. The (Cartesian) space-time coordinates are $x^\mu = (ct, x, y, z)$, Greek indices run from 0—the "temporal" coordinate—to 3, while Roman indices go from 1 to 3—the "spatial" coordinates. We use the metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{17}$$

and the Einstein convention (sum repeated indices). The energy density u, the energy flux **S**, the momentum density **g**, and the stress tensor $\overleftrightarrow{T}$ go together to make the **stress-energy tensor**:

$$\Theta^{\mu\nu} = \begin{pmatrix} u & [\mathbf{S}/c] \\ [c\mathbf{g}] & [-\overleftrightarrow{T}] \end{pmatrix} \tag{18}$$

This is entirely general—in other contexts u, **S**, **g**, and $\overleftrightarrow{T}$ will not have their electromagnetic form [Eqs. (7)–(10)]. If the stress-energy tensor is divergenceless:

$$\partial_\mu \Theta^{\mu\nu} = 0 \tag{19}$$

then

$$p^\mu \equiv \int \Theta^{0\mu}\, d^3\mathbf{r} \tag{20}$$

transforms as a four-vector [this is sometimes called "von Laue's theorem" (Refs. 24, 26, and 7)], and the total energy and momentum ($E = cp^0$ and $\mathbf{p}$) are conserved. If the stress tensor is *symmetric* ($\Theta^{\nu\mu} = \Theta^{\mu\nu}$), then angular momentum is also conserved (Ref. 11). In a well-formulated theory the *complete* stress-energy tensor is always divergenceless and symmetric, but this may not be true for individual portions of it, such as the electromagnetic contribution alone.

2. *Electrodynamics*

The charge and current densities combine to form a four-vector:

$$J^\mu = (c\rho, J_x, J_y, J_z); \tag{21}$$

the fields constitute an antisymmetric tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}; \tag{22}$$

and the potentials make a four-vector

$$A^\mu = (V/c, A_x, A_y, A_z). \tag{23}$$

The inhomogeneous Maxwell equations read

$$\partial_\mu F^{\mu\nu} = \frac{1}{\epsilon_0 c} J^\nu \tag{24}$$

(where ∂_μ is short for $\partial/\partial x^\mu$). The homogeneous Maxwell equations are enforced by the potential representation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{25}$$

The electromagnetic stress-energy tensor is:

$$T^{\mu\nu} = \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ cg_x & -T_{xx} & -T_{xy} & -T_{xz} \\ cg_y & -T_{yx} & -T_{yy} & -T_{yz} \\ cg_z & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix}. \tag{26}$$

In view of Eq. (16), $T^{\mu\nu}$ is symmetric; in terms of the fields:

$$T^{\mu\nu} = \epsilon_0 \left(g^{\mu\kappa} F_{\kappa\lambda} F^{\lambda\nu} + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \right). \tag{27}$$

The continuity equation becomes the statement that J^μ is divergenceless:

$$\partial_\mu J^\mu = 0. \tag{28}$$

The electromagnetic stress-energy tensor is not by itself divergenceless; from Maxwell's equations it follows that

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} F^{\kappa\nu} J_\kappa. \tag{29}$$

Ordinarily, therefore, electromagnetic energy and momentum

$$p^0_{\text{em}} = \int T^{00}\, d^3\mathbf{r} \quad \text{and} \quad p^i_{\text{em}} = \int T^{0i}\, d^3\mathbf{r}, \tag{30}$$

do not constitute a four-vector, and they are not conserved. However, if $J_\kappa = 0$ (for instance, in empty space) then p^μ_{em} *is* a conserved four-vector. And (as always) the *complete* stress-energy tensor,

$$\Theta^{\mu\nu} = T^{\mu\nu} + \Theta^{\mu\nu}_{\text{o}}, \tag{31}$$

(where $\Theta^{\mu\nu}_{\text{o}}$ is the non-electromagnetic contribution) *is* divergenceless (and symmetric).

The energy/momentum radiated by a point charge q is

$$\frac{dp^\mu}{dt} = \frac{\mu_0 q^2}{6\pi c^3} (\alpha^\nu \alpha_\nu) \eta^\mu, \tag{32}$$

where $\eta^\mu \equiv dx^\mu/d\tau$ is the four-velocity and $\alpha^\mu \equiv d\eta^\mu/d\tau$ is the four-acceleration ($d\tau$ is the proper time).

III. MOMENTUM AND VECTOR POTENTIAL

For a localized configuration the total momentum in the fields is

$$\mathbf{p} = \int \mathbf{g}\, d^3\mathbf{r} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{r}. \tag{33}$$

[I assume all fields go to zero sufficiently rapidly at infinity that the integrals converge, and surface terms can be neglected. It is notoriously dangerous to speak of the momentum (or energy) of a configuration that is not localized in space, and when I refer to a "uniform" field this should always be interpreted to mean *locally* uniform, but going to zero at infinity.] In the *static* case two equivalent expressions can be obtained, by writing either $\mathbf{E}$ or $\mathbf{B}$ in terms of the potentials ($\mathbf{E} = -\nabla V$, or $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = 0$) (Refs. 51, 47, 38, and 61):

$$\mathbf{p} = \int \rho \mathbf{A}\, d^3\mathbf{r}, \tag{34}$$

$$\mathbf{p} = \frac{1}{c^2} \int V \mathbf{J}\, d^3\mathbf{r}. \tag{35}$$

In particular, the electromagnetic momentum of a stationary point charge q, in a magnetic field represented by the vector potential $\mathbf{A}$, is

$$\mathbf{p} = q\mathbf{A}. \tag{36}$$

This suggests that $\mathbf{A}$ can be interpreted as "potential momentum" per unit charge, just as V is potential *energy* per unit charge.

The association between momentum and vector potential goes back to Maxwell, who called $\mathbf{A}$ "electromagnetic momentum" (Ref. 41; p. 481) and later "electrokinetic momentum" (Ref. 10; Art. 590), and Thomson (Ref. 21). But the idea did not catch on; *any* physical interpretation of $\mathbf{A}$ was disparaged by Heaviside and Hertz (Refs. 34 and 36), who regarded $\mathbf{A}$ as a purely mathematical device. So generations of teachers were left with no good answer to their students' persistent question: "What does the vector potential represent, physically?" Few were satisfied by the safe but unilluminating response, "It is that function whose curl is $\mathbf{B}$" (Ref. 39). From time to time the connection to momentum was rediscovered [by Calkin (Ref. 35), for example], but it was not widely recognized until Konopinski's pivotal paper (Ref. 40). Konopinski was apparently unaware of the historical background, which was supplied by Gingras (Ref. 37).

Many modern authors follow Konopinski's lead, culminating in what remains (to my mind) the definitive discussion by Semon and Taylor (Ref. 42).

An obvious objection is that the vector potential is not gauge invariant, and different choices yield different momenta. Semon and Taylor point out that generalized momentum itself is ambiguous: canonical momentum, for instance, does not always coincide with ordinary ("kinetic") momentum. In any event, Eq. (34) holds only for *static* fields. But in truth, the same objections could be raised against the interpretation of V as potential energy per unit charge. [A point charge at rest can be represented by the potentials $V(\mathbf{r}, t) = 0$, $\mathbf{A}(\mathbf{r}, t) = -(1/4\pi\epsilon_0)qt/r^2\,\hat{\mathbf{r}}$, and while no physicist in her right mind would choose to do so, the fact remains that the physical meaning of V depends on the gauge. Moreover, if the fields are time dependent, the work done to move a charge is no longer $q\Delta V$ in *any* gauge.]

Another way to get at the association between momentum and potential is afforded by the Lagrangian formulation of electrodynamics. For a nonrelativistic particle of mass m and charge q, moving with velocity $\mathbf{v}$ through fields described by the potentials $V(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ (Ref. 5; Sec. 4.9)

$$L(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2}mv^2 + q\mathbf{v}\cdot\mathbf{A} - qV. \tag{37}$$

The generalized momentum $(p_i = \delta L/\delta \dot{q}_i)$ (Ref. 20) is

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}, \tag{38}$$

the sum of a purely kinetic part ($m\mathbf{v}$) and an electromagnetic part ($q\mathbf{A}$). The Hamiltonian $(\mathbf{p}\cdot\mathbf{v} - L)$ is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV. \tag{39}$$

It differs from the *free* particle Hamiltonian ($H = p^2/2m$) by the substitution

$$\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}, \quad H \rightarrow H - qV. \tag{40}$$

This is the so-called "minimal coupling" rule—an efficient device for constructing the Hamiltonian of a charged particle in the presence of electromagnetic fields. It is equivalent to the Lorentz force law and is especially useful in quantum mechanics (Ref. 14, Sec. 6.8). In the relativistic theory (Ref. 6, Sec. 12.1) the generalized four-momentum is

$$p^\mu = m\eta^\mu + qA^\mu, \tag{41}$$

and minimal coupling becomes (Ref. 3; p. 360)

$$p^\mu \rightarrow p^\mu - qA^\mu. \tag{42}$$

Thus, relativity reinforces the notion that if V is (potential) energy per unit charge, then $\mathbf{A}$ is (potential) momentum per unit charge, drawing a parallel between the four-vectors $p^\mu = (E/c, \mathbf{p})$ and $A^\mu = (V/c, \mathbf{A})$.

IV. HIDDEN MOMENTUM

Even purely *static* electromagnetic fields can harbor momentum (Eqs. (33)–(35)). Configurations that have been studied include

- An ideal (point) magnetic dipole $\mathbf{m}$ in an external electric field $\mathbf{E}$ (Ref. 51):

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{E}\times\mathbf{m}). \tag{43}$$

[This is for a *conventional* magnetic dipole—a tiny current loop. If the dipole is made of hypothetical magnetic monopoles, the momentum is zero (Ref. 52).]

- An ideal (point) electric dipole $\mathbf{p}_e$ in an external magnetic field $\mathbf{B}$ (Ref. 61):

$$\mathbf{p} = \frac{1}{2}(\mathbf{B}\times\mathbf{p}_e). \tag{44}$$

- A sphere (radius R) carrying a surface charge $\sigma = k\cos\theta$, where k is a constant and θ is the polar angle with respect to the z axis (its electric dipole moment is $\mathbf{p}_e = 4/3\pi R^3 k\,\hat{\mathbf{z}}$). It also carries a surface current $\mathbf{K} = k'\sin\theta'\hat{\phi}'$, where k' is another constant and θ', ϕ' are the polar and aximuthal angles with respect to the z' axis (its magnetic dipole moment is $\mathbf{m} = 4/3\pi R^3 k'\,\hat{\mathbf{z}}'$). The momentum in the fields is (Ref. 63)

$$\mathbf{p} = \frac{\mu_0}{4\pi R^3}(\mathbf{m}\times\mathbf{p}_e) \tag{45}$$

- A charged parallel-plate capacitor (field $\mathbf{E}$, volume $\mathcal{V}$) in a uniform magnetic field $\mathbf{B}$ (Refs. 61, 80, 57, and 43):

$$\mathbf{p} = \frac{1}{2}\epsilon_0(\mathbf{E}\times\mathbf{B})\mathcal{V}. \tag{46}$$

[Ordinarily, electromagnetic momentum (like electromagnetic energy), being quadratic in the fields, does not obey the superposition principle (that is, the momentum of a composite system is not the sum of the momenta of its parts, considered in isolation). However, if static charges are placed in an external magnetic field, the momentum is linear in the *electric* field they produce, and hence the total momentum *is* the sum of the individual momenta. That's how McDonald (Ref. 61) discovered the (surprising) factor of 1/2 in Eq. (46), which is due to momentum in the fringing fields.]

Now, there is a very general theorem in relativistic field theory that says

> **If the center of energy of a closed system is at rest, then the total momentum is zero.**

["Center of energy" is the relativistic generalization of center of mass, but it takes account of *all* forms of energy, not just rest energy: $\int \mathbf{r}u\,d^3\mathbf{r}/\int u\,d^3\mathbf{r}$.] This certainly seems reasonable; a heuristic argument is given by Calkin (Ref. 47), and a more formal proof by Coleman and Van Vleck (Ref. 48). In the configurations described above the center of energy is clearly at rest, so if there is momentum in the fields there must be compensating *non*-electromagnetic momentum somewhere else in the system. But it is far from obvious where this "hidden momentum" resides, or what its nature might be.

Curiously, the phenomenon of hidden momentum was not noticed until the work of Shockley and James (Ref. 64) and Costa de Beauregard (Ref. 50), in 1967. It was picked up immediately by Haus and Penfield (Ref. 53), Coleman and Van Vleck (Ref. 48), Furry (Ref. 51), and eventually by many others (Refs. 47, 65, 49, and 54). Indeed, the subject remains an active area of research to this day (Refs. 55, 58, and 59).

The simplest model for hidden momentum was suggested by Calkin (Ref. 47) (or Ref. 2; Example 12.12); it consists of a steady current loop in an external electric field. The current is treated as a stream of free charges, speeding up and slowing down in response to the field. [Because the *current* is the same all around the loop, in segments where the charges are moving more rapidly they are farther apart.] Each charge carries a (relativistic) mechanical momentum $(mv/\sqrt{1-(v/c)^2})$, and—even though the loop is not moving and the current is steady—these momenta add up to a total that exactly cancels the electromagnetic momentum.

Others have noted that this is an artificial model for the current, and Vaidman (Ref. 65) considered two more realistic models, an incompressible fluid, and a metal wire. The former carries mechanical momentum because of the remarkable (relativistic) fact that a moving fluid under pressure has "extra" momentum (Ref. 56), whereas the latter, because of induced charges on the surface of the wire, has no momentum in the fields (and no hidden momentum to cancel it). [This applies as well to the examples above; to be safe, we assume that all charges are glued to nonconductors, and the magnetic fields are produced by charged nonconductors in motion (Refs. 51, 65, and 57).]

Hidden momentum has nothing to do, really, with electrodynamics, except that it was first discovered in this context. The name itself is perhaps unfortunate, since it *sounds* mysterious, and a definitive characterization of hidden momentum remains elusive. This much seems clear: it is mechanical, relativistic, and occurs in systems that are at rest, but have internally moving parts. [Actually, there is no reason the system has to be at rest, but the phenomenon is much more striking in that case, and there has not been much discussion of hidden momentum in moving configurations. Similarly, there exists in principle hidden *angular* momentum, but since there is no rotational analog to the center-of-energy theorem, it is less intriguing—indeed, many examples are known in which nothing is rotating, and yet the fields carry angular momentum with no compensating hidden angular momentum. The extreme example is the Thomson dipole, consisting of a magnetic monopole and an electric charge (Ref. 46).] It is "hidden" only in the sense that it is surprising and unexpected, but it is perfectly genuine momentum. [To this day some authors remain skeptical (Ref. 44); Mansuripur (Refs. 126 and 60) calls hidden momentum "absurdity."]

Not every case of momentum in static fields involves hidden momentum. A long coaxial cable connected to a battery at one end and a resistor at the other carries electromagnetic momentum (Ref. 2; Example 8.3), but no hidden momentum. In this case the center of energy is *not* at rest; energy is flowing from the battery to the resistor, and the associated momentum is precisely the momentum in the fields (Refs. 45 and 62).

V. MOMENTUM AND MASS

The energy in the electric field of a uniformly charged stationary spherical shell, of radius R and charge Q, is (Ref. 2; Example 2.8):

$$E = \frac{Q^2}{8\pi\epsilon_0 R}. \tag{47}$$

According to Einstein's formula ($E = mc^2$) this means there is an electromagnetic contribution to its mass, in the amount

$$m_{\text{em}1} = \frac{Q^2}{8\pi\epsilon_0 Rc^2} \tag{48}$$

If the sphere is now set in motion, at a constant nonrelativistic velocity $\mathbf{v}$, the momentum in its electromagnetic fields is (Refs. 18, 74, 69, 75, and 70)

$$\mathbf{p} = \frac{2Q^2}{3Rc^2}\mathbf{v} \tag{49}$$

from which we infer that there is an electromagnetic contribution to its mass in the amount

$$m_{\text{em}2} = \frac{2Q^2}{3Rc^2} = \frac{4}{3}m_{\text{em}1}. \tag{50}$$

The momentum-derived mass is greater than the energy-derived mass, by an infamous factor of 4/3 (Refs. 73, 71, and 72). This ratio holds for all spherically-symmetrical charge configurations; other geometries yield different factors (Ref. 70).

The underlying source of the discrepancy has been known for over a century. Poincaré (Ref. 76) pointed out that a charged sphere is unstable (it would explode, from the mutual repulsion of its parts), unless some *other* force is provided, to hold it together. This stabilizing force (whatever its nature) has come to be known as "Poincaré stress," and it too contributes to the energy and momentum of the object. When the two contributions are combined, the inconsistency disappears. [A closely related problem is "Rindler's paradox" (Ref. 77).]

More formally, the problem is that the electromagnetic stress-energy tensor is not divergenceless in the presence of charge and current [Eq. (29)], and as a result the integral

$$p^\mu_{\text{em}} = \int T^{0\mu}\, d^3\mathbf{r} \tag{51}$$

does not constitute a four-vector. It is only the *complete* stress-energy tensor (which in this instance would include a contribution from the Poincaré stress) that is divergenceless, and *its* integral does yield a genuine (and conserved) four-vector (Eq. (20)).

In the early years of the 20th century, when Abraham (Ref. 66), Lorentz (Ref. 9), Schott (Ref. 19), and others dreamed of producing a purely electromagnetic model of the electron (Ref. 79), there was a sense that the electromagnetic fields "ought" to yield an energy-momentum four-vector all by themselves, and this notion has persisted—even in the face of Poincaré's (to my mind decisive) argument that they should *not*. Rohrlich (Refs. 78 and 18) proposed to retain the equation for electromagnetic energy in the particle's rest frame, but define the electromagnetic momentum by Lorentz transformation. This makes p^μ_{em} a four-vector by construction, but it means Eq. (1) is no longer applicable in the presence of charges and currents. [In empty space, where $J^\mu = 0$, the electromagnetic stress-energy tensor *is* divergenceless (Eq. (29)), and the problem does not arise. Thus, the application of Eq. (1) to electromagnetic waves in vacuum has not, to my knowledge, been challenged.] Rohrlich's suggestion was criticized by Tangherlini (Ref. 81), Boyer (Ref. 67), and others (Ref. 68), but it has left a residue of confusion as to the "correct" expression for electromagnetic momentum (Ref. 80).

VI. MOMENTUM IN MATTER

In the presence of materials subject to polarization ($\mathbf{P}$) and magnetization ($\mathbf{M}$) it is convenient to express the laws of electrodynamics in terms of *free* charges and *free* currents, since these are the ones we directly control (Ref. 2; Sec. 7.3.5):

$$\rho = \rho_f + \rho_b, \quad \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p. \tag{52}$$

The "bound charge," "bound current," and "polarization current" are

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \mathbf{J}_b = \nabla \times \mathbf{M}, \quad \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \tag{53}$$

Introducing the auxiliary fields

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{54}$$

Maxwell's equations become

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{55}$$

If the medium is *linear* (as we shall assume from now on), **D** and **H** are related to **E** and **B** by the "constitutive relations"

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}, \tag{56}$$

where ϵ is the permittivity and μ is the permeability of the material. [In *non*-linear media—such as ferromagnets—the work done depends not only on the initial and final states of the system but also on *how* it was carried from one to the other, so the whole notion of "stored energy" loses its meaning (Ref. 17; Secs. 4.10 and 10.10).]

Although the original formulas for electromagnetic energy (Eq. (7)) and energy flux (Eq. (9)) are still perfectly correct, they are not very useful in this context: $(\epsilon_0/2)E^2$, for instance, is the work it would take to bring in all the charges (free and bound) from infinity and nail them down in their final locations—but it does not include the work required to stretch all the atomic "springs" to which the bound charges are attached. A more useful quantity is the work done on the free charges alone, as we bring them in from infinity, with the bound charges (and the springs) responding however they do. The resulting electromagnetic *energy density in matter* is (Ref. 2; Problem 8.15), (Ref. 6; Sec. 6.7), (Refs. 8, 98, 113, and 114)

$$u_m = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}), \tag{57}$$

and the corresponding flux (the "Poynting vector") is

$$\mathbf{S}_m = \frac{1}{\mu}(\mathbf{E}\times\mathbf{B}) = (\mathbf{E}\times\mathbf{H}). \tag{58}$$

In the same spirit we might calculate the force per unit volume on the *free* charges in the material:

$$f_m = \rho_f \mathbf{E} + \mathbf{J}_f \times \mathbf{B} = \nabla \cdot \overleftrightarrow{\mathrm{T}}_m - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}), \tag{59}$$

where (Ref. 10; Art. 641), (Ref. 13; Chapter X)

$$(T_m)_{ij} = \epsilon\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$
$$= \left(E_iD_j - \frac{1}{2}\delta_{ij}\mathbf{E}\cdot\mathbf{D}\right) + \left(B_iH_j - \frac{1}{2}\delta_{ij}\mathbf{B}\cdot\mathbf{H}\right). \tag{60}$$

Comparison with Eq. (15) invites us to interpret

$$\mathbf{g}_M \equiv (\mathbf{D}\times\mathbf{B}) \tag{61}$$

as the electromagnetic momentum in a linear medium. This was Minkowski's proposal (Ref. 129).

However, the resulting electromagnetic stress-energy tensor has the form

$$(T_M)^{\mu\nu} = \begin{pmatrix} u_m & [\mathbf{S}_m/c] \\ [c\mathbf{g}_M] & [-\overleftrightarrow{\mathrm{T}}_M] \end{pmatrix}, \tag{62}$$

and Abraham pointed out that because $\mathbf{S}_m/c \neq c\mathbf{g}_M$ Minkowski's tensor is not symmetric, and hence does not conserve angular momentum. He suggested instead that the electromagnetic momentum in matter is (Refs. 82, 83, and 84).

$$\mathbf{g}_A \equiv \frac{1}{c^2}(\mathbf{E}\times\mathbf{H}), \tag{63}$$

while u_m, $\mathbf{S}_m$, and $\overleftrightarrow{\mathrm{T}}_m$ are unchanged (Ref. 11; pp. 204–205). This entails replacing Eq. (59) by

$$\nabla \cdot \overleftrightarrow{\mathrm{T}}_m - \frac{\partial}{\partial t}\mathbf{g}_A = (\rho_f \mathbf{E} + \mathbf{J}_f \times \mathbf{B}) + \mathbf{f}_\mathrm{A}, \tag{64}$$

with the extra "Abraham force" density (Refs. 94 and 128)

$$\mathbf{f}_\mathrm{A} = \left(1 - \frac{1}{n^2}\right)\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}), \tag{65}$$

where $n \equiv \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ is the index of refraction of the medium. [Most studies concentrate on dielectric materials, with $\mu = \mu_0$; some treat purely *magnetic* materials ($\epsilon = \epsilon_0$). I will keep the discussion general, whenever possible, though I do assume that dispersion can be neglected (i.e., ϵ and μ do not depend significantly on frequency).]

Thus, began a debate that has raged for more than a century, between Minkowski's momentum and Abraham's. Scores of theoretical papers have claimed to settle the issue in favor of one or the other; experiments have been performed with unambiguous (but contradictory) results. In recent years the dispute has been particularly intense, because of the critical importance of optical forces in nanotechnology (Refs. 85, 97, 106, and 29).

How would one go about determining the momentum of the electromagnetic fields in a medium? The obvious test would be the pressure of light on the interface between two transparent media, or on a mirror embedded in the material. Conservation of momentum should dictate the correct formula. Imagine a wave packet in vacuum, with total energy U and momentum p (Fig. 1(a)),

$$U = \left(\frac{1}{2}\epsilon_0 E^2\right)AL, \quad p = \left(\epsilon_0 \frac{E^2}{2c}\right)AL = \frac{U}{c} \tag{66}$$

(here E is the amplitude of the electric field, E/c is the amplitude of the magnetic field, and we average over a full cycle).

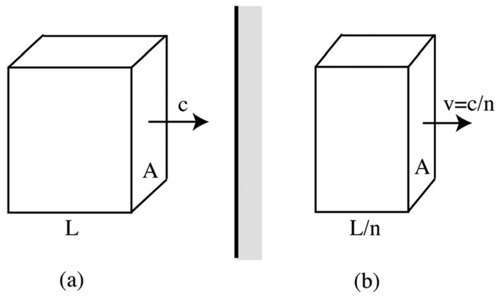

Fig. 1. A pulse of light in vacuum (a) and in a medium (b).

The same packet (with the same energy) travels more slowly ($v = c/n$) in a transparent medium (Fig. 1(b)):

$$U = \left(\frac{1}{2}\epsilon E^2\right)AL\frac{v}{c}, \quad \begin{cases} p_M = \left(\epsilon\frac{E^2}{2v}\right)AL\frac{v}{c} = \frac{U}{v} = np, \\ p_A = \left(\frac{1}{c^2\mu}\frac{E^2}{2v}\right)AL\frac{v}{c} = \frac{vU}{c^2} = \frac{p}{n}. \end{cases} \tag{67}$$

The momentum of the packet is *in*creased (by a factor of n), according to Minkowski, whereas according to Abraham it *de*creases by the same factor (Refs. 137, 117, 135, and 89). It should be very easy to discriminate.

Example 1. In 1953 Balazs (Ref. 87) proposed the following (now classic) thought experiment. Start with a bar of transparent material, its two ends coated to prevent reflections. Imagine two identical pulses of light, each of energy U (the pulses are very short, compared to the length of the bar). Pulse (1) passes *through* the bar (Fig. 2); pulse (2) passes just outside the bar (Fig. 3). The bar itself is free to move, but relatively heavy ($Mc^2 \gg U$), so it can absorb momentum from pulse (1) without acquiring significant kinetic energy.

Because there are no external forces, the center of energy (X) of the system (pulse plus bar) moves at a constant rate, and at any given time is at the same location in both cases. During the time τ that pulse (1) spends inside the bar, the center of energy of system (2) moves a distance

$$\Delta X_2 = \frac{1}{U + Mc^2}[U(c\tau)]. \tag{68}$$

Pulse (1) does not move as far (because it travels at a reduced speed in the medium), so the bar itself must move a (small) distance ϵ to make up for it.

$$\Delta X_1 = \frac{1}{U + Mc^2}[Mc^2(\epsilon) + U(w + L + \epsilon)]. \tag{69}$$

Equating the two ($\Delta X_2 = \Delta X_1$), we find that

$$\epsilon = \frac{U}{Mc^2 + U}(c\tau - w - L) \approx \frac{U}{Mc^2}(c - v)\tau \tag{70}$$

(note that $v\tau \approx w + L$). Evidently the *momentum* of the bar, during the time pulse (1) is inside, is

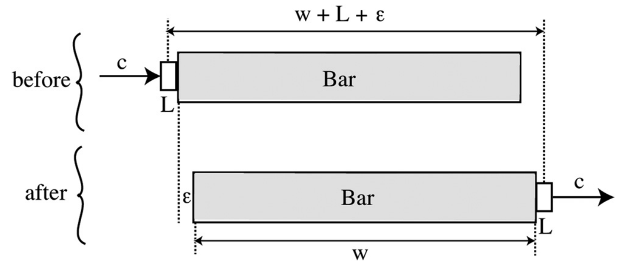

Fig. 2. Pulse 1.

$$p_{\text{bar}} = \frac{M\epsilon}{\tau} = \frac{U}{c}\left(1 - \frac{v}{c}\right). \tag{71}$$

The total momentum before the pulse enters the bar (and also after it exits) is $p = U/c$; conservation requires that this equal the momentum of the pulse while inside the bar (p_{in}) plus the momentum of the bar itself,

$$p = p_{\text{in}} + p_{\text{bar}}, \tag{72}$$

and hence

$$p_{\text{in}} = p\frac{v}{c} = \frac{p}{n}. \tag{73}$$

Score one for Abraham (compare Eq. (67)).

Example 2. Imagine a monochromatic plane wave, propagating in the z direction through a homogeneous linear material, that encounters a perfectly reflecting surface at $z = 0$. The incident and reflected fields are (Ref. 2; Sec. 9.3)

$$\begin{aligned} \mathbf{E}^I(z,t) &= E_0\cos(kz - \omega t)\,\hat{\mathbf{x}}, \\ \mathbf{B}^I(z,t) &= \frac{E_0}{v}\cos(kz - \omega t)\,\hat{\mathbf{y}} \\ \mathbf{E}^R(z,t) &= -E_0\cos(-kz - \omega t)\,\hat{\mathbf{x}}, \\ \mathbf{B}^R(z,t) &= \frac{E_0}{v}\cos(-kz - \omega t)\,\hat{\mathbf{y}}, \end{aligned}$$

(for $z \le 0$, and zero for $z > 0$). The discontinuity in $\mathbf{B}$ determines the surface current:

$$\mathbf{K} = \frac{2E_0}{\mu v}\cos(\omega t)\,\hat{\mathbf{x}}. \tag{74}$$

The force on an area A of surface is

$$\begin{aligned} \mathbf{F} &= [\sigma\mathbf{E} + (\mathbf{K} \times \mathbf{B})]A = \frac{2E_0^2 A}{\mu v^2}\cos^2(\omega t)\,\hat{\mathbf{z}} \\ &= 2\epsilon E_0^2 A\cos^2(\omega t)\,\hat{\mathbf{z}} \end{aligned} \tag{75}$$

(since $\mathbf{B}$ is discontinuous at $z = 0$, we use the average). This is the momentum per unit time imparted to the mirror (no *energy* is delivered, if we hold the mirror stationary). Meanwhile, the momentum per unit time dumped by the electromagnetic wave, as it reverses direction, is

$$\frac{d\mathbf{p}}{dt} = \begin{cases} 2\left(\epsilon\frac{E_0^2\cos^2(\omega t)}{v}\right)Av = 2\epsilon E_0^2 A\cos^2(\omega t)\,\hat{\mathbf{z}} & \text{(Minkowski)} \\ 2\left(\frac{1}{c^2\mu}\frac{E_0^2\cos^2(\omega t)}{v}\right)Av = \frac{2}{n^2}\epsilon E_0^2 A\cos^2(\omega t)\,\hat{\mathbf{z}} & \text{(Abraham).} \end{cases} \tag{76}$$

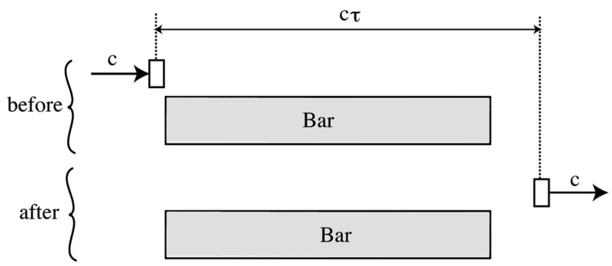

Fig. 3. Pulse 2.

Point Minkowski. (For related examples see (Refs. 120, 121, 122, 123, 108, 125, and 102).)

This experiment was performed by Jones and collaborators (Refs. 109, 110, and 111), and the Minkowski prediction was confirmed. [Actually, Jones measured the force on the mirror as a function of n, for various liquids; Minkowski says the force is proportional to n, whereas Abraham says it should go like $1/n$. Mansuripur (Ref. 122) has argued that the experiment would have supported Abraham had they used a mirror that did not reverse the phase of the reflected wave (a "perfect magnetic conductor," instead of a perfect electric conductor), but Kemp and Grzegorczyk (Ref. 112) show that even in that case the Minkowski prediction is ultimately sustained.]

Other models have been explored (Refs. 86, 96, and 135), including ones involving diffraction (see Example 3 below), and even purely static configurations (for which, however, hidden momentum may have to be considered) (Refs. 116, 57, and 95). Some seem to favor Minkowski, others Abraham. Beginning in the late 1960's (Refs. 15, 4, 105, 131, and 93) something approaching a consensus emerged: Both the Minkowski momentum *and* the Abraham momentum are "correct," but they speak to different issues, and it is largely a matter of taste which of the two (or perhaps even one of the other candidates that have from time to time been proposed (Refs. 103, 131, 100, 124, and 136) one identifies as the "true" electromagnetic momentum. The essential point goes back to Poincaré (Ref. 76): in the presence of matter the electromagnetic stress-energy tensor by itself is not conserved (divergenceless). Only the *total* stress-energy tensor carries unambiguous physical significance, and how one apportions it between an "electromagnetic" part and a "matter" part depends on context and convenience. Minkowski did it one way, Abraham another; they simply regard different portions of the total as "electromagnetic" (Refs. 130, 142, 128, and 134). Except in vacuum, "electromagnetic momentum" by itself is an intrinsically ambiguous notion.

For example, when light passes through matter it exerts forces on the charges, setting them in motion, and delivering momentum to the medium. Since this is associated with the wave, it is not unreasonable to include some or all of it in the electromagnetic momentum, even though it is purely mechanical in nature. But figuring out exactly how and where this momentum is located can be very tricky. For instance, in Example 2 momentum is imparted not just to the mirror, but also to the dielectric in front of it, so the fact that Minkowski's momentum balances that of the mirror is to some extent fortuitous. As Baxter and Loudon put it (Ref. 91), "the total momentum transfer has the Abraham value …the [dielectric] liquid takes up the difference between the Abraham and Minkowski …momenta, which is eventually transferred to its container."

One would like to write down, once and for all, the complete and correct *total* stress-energy tensor—electromagnetic plus mechanical. Unfortunately, this depends on the detailed nature of the material (Refs. 127 and 132), and in realistic theories can be forbiddingly complicated.

What exactly do the two momenta represent, physically, and why is it that some contexts seem to favor the one or the other (Ref. 133)? Recently, several authors (Refs. 90, 104, 117, 107, 91, and 101) and especially Barnett (Refs. 88 and 89) have noted that there is a parallel ambiguity between the "kinetic" momentum (mv) of a particle and its "canonical" momentum. Abraham's field momentum is associated with the former, and Minkowski's with the latter; the conserved *total* is

$$\mathbf{p}_{\text{total}} = \mathbf{p}_{\text{kinetic}} + \int \mathbf{g}_A\, d^3\mathbf{r} = \mathbf{p}_{\text{canonical}} + \int \mathbf{g}_M\, d^3\mathbf{r}. \tag{77}$$

You can use either the Abraham momentum or the Minkowski momentum for the fields, as long as you combine it with the appropriate momentum for the particles. In Example 1 we used the kinetic momentum of the bar (Eq. (71)), so it was appropriate in this case to use the Abraham momentum for the fields. Baxter and Loudon (Ref. (91)) associate "Abraham …momentum with the motion of a dielectric specimen as a whole and …Minkowski momentum with the motion of objects embedded in the dielectric."

There are several excellent reviews of the entire controversy [and at least one dissertation (Ref. 92)]: Brevik (Ref. 93) offers a detailed survey of the relevant experiments, Pfeifer *et al.* (Ref. 132) is the most comprehensive, Milonni and Boyd (Ref. 128) and Baxter and Loudon (Ref. 91) are the most up-to-date (and in my opinion the clearest and most accessible). But the final word on this vexed subject has certainly not been written—indeed, the frequency of papers continues to grow.

VII. MOMENTUM OF PHOTONS

In 1900 Max Planck (Ref. 143) proposed that electromagnetic waves come in little squirts ("quanta"), with energy

$$E = h\nu, \tag{78}$$

where ν is the frequency and h is the Planck's constant (empirically, $h = 6.626 \times 10^{-34}$ J s). Planck did not pretend to know *why* the energy is quantized—he assumed it had something to do with the emission process. In 1905 Albert Einstein (Ref. 141) reinterpreted Planck's quanta as "particles" of light (Gilbert Lewis suggested the name "photon" in 1926). Einstein's idea was widely ridiculed (Ref. 12; Secs. 18a and 19f) until 1923, when Compton (Ref. 140) accounted for the change in wavelength of light scattered from a charged particle by treating the light as a massless particle with energy given by Planck's formula and momentum dictated by the relativistic invariant $E^2 - p^2c^2 = m^2c^4$:

$$p = \frac{E}{c} = \frac{h\nu}{c}. \tag{79}$$

For an authoritative history see Pais's biography of Einstein, "Subtle is the Lord" (Ref. 12).

The photon picture offers a more tangible way to think about electromagnetic momentum: Instead of the rather abstract notion of momentum stored in fields, it is simply the total momentum of the photons present. In particular, the photon picture illuminates the Abraham–Minkowski controversy about electromagnetic momentum in a dielectric medium. Is the momentum of a photon in matter p/n, as Abraham would have it, or pn, as Minkowski requires (Refs. 104, 138, 88, and 89)?

$$\begin{cases} p_{\rm A} = \dfrac{1}{n}\left(\dfrac{h\nu}{c}\right) & \text{(Abraham)} \\ p_{\rm M} = n\left(\dfrac{h\nu}{c}\right) & \text{(Minkowski)} \end{cases} \tag{80}$$

Some of the classic examples can be presented more cleanly in this language (Refs. 119 and 88), and it is essential in explicitly quantum mechanical arguments such as the following (Refs. 139, 142, and 91).

Example 3. The size of the central maximum in single-slit diffraction can be estimated using the uncertainty principle, $\Delta p \Delta x \geq \hbar/2$. Here Δx is the width of the slit, w, so

$$\Delta p \geq \hbar/2w. \tag{81}$$

The resulting angular spread of the beam is

$$\Delta\theta = 2\frac{\Delta p}{p} = \frac{c}{2\pi\nu w}. \tag{82}$$

Now suppose the whole system is emmersed in a transparent medium with index of refraction n. Equation (81) is still the uncertainty in p, but p itself is now given by Eq. (80). Evidently the new angular divergence is

$$\Delta\theta' = \begin{cases} n\,\Delta\theta & \text{(Abraham)} \\ \dfrac{1}{n}\,\Delta\theta & \text{(Minkowski)} \end{cases} \tag{83}$$

Does the pattern spread out, as Abraham would have it, or does it contract, as Minkowski predicts? In point of fact it contracts (that is why the oil-immersion microscope has better resolution).

It is no surprise that Minkowski wins this one. The momentum that goes in the uncertainty principle is *canonical* momentum, and we have seen (Eq. (77)) that canonical momentum is associated with the Minkowski form.

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C.V. Boys' Rainbow Cup. This thin film interference demonstration is due to C.V. Boys, better known for developing the method of making extremely thin quartz suspension fibers. The plastic wind shield is removed, and a thin film is formed across the open top of the hemisphere (whose interior is painted flat black) by dipping the apparatus into a soap solution. It is then placed on a rotator and spun. The film is viewed by reflected light, and a series of concentric, colored interference rings is observed. The film is thinnest at the center, and if the rotation rate is large enough, the center is black. This apparatus is listed at $14.50 in the 1950 Cenco catalogue, and is in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)