SPACES OF ULTRAFILTERS

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Filters

01° Let X be any set. By a *filter* on X, we mean a nonempty family $\mathcal{F}$ of subsets of X which meets the following conditions:

(1) $\emptyset \notin \mathcal{F}$
(2) $F \in \mathcal{F},\ G \in \mathcal{F} \Longrightarrow F \cap G \in \mathcal{F}$
(2) $F \in \mathcal{F},\ F \subseteq H \Longrightarrow H \in \mathcal{F}$

where F, G, and H are any subsets of X.

02° It may happen that a nonempty family $\mathcal{F}_o$ of subsets of X meets conditions (1) and (2) but (perhaps) not (3). In such a case, we introduce the family $\mathcal{F}$ consisting of all subsets G of X such that there is some F in $\mathcal{F}$ for which $F \subseteq G$. Obviously, $\mathcal{F}$ is a filter on X, as it meets not only conditions (1) and (2) but also (3). We say that $\mathcal{F}_o$ *generates* $\mathcal{F}$.

03° For instance, we may select a member ξ of X, then take $\mathcal{F}_o$ to be the family consisting of the singleton $\{\xi\}$. In such a case, we refer to the filter generated by $\mathcal{F}_o$ as the *principal* filter on X defined by ξ. We denote it by $\mathcal{P}_\xi$.

04° Let $\mathcal{F}$ be a filter on X. Let A and B be subsets of X such that $A \cup B \in \mathcal{F}$. We contend that if $B \notin \mathcal{F}$ then there is a filter $\mathcal{G}$ on X such that:

$$\mathcal{F} \cup \{A\} \subseteq \mathcal{G}$$

To prove the contention, we argue as follows. Let us form the family $\mathcal{G}_o$ of subsets of X of the form $F \cap A$, where F runs through $\mathcal{F}$. Obviously, $\mathcal{G}_o$ meets condition (2). Moreover, if there were some F in $\mathcal{F}$ for which $F \cap A = \emptyset$ then $F \cap (A \cup B) = F \cap B$, so that B would be in $\mathcal{F}$, a contradiction. Consequently, $\mathcal{G}_o$ meets condition (1). Now we need only take $\mathcal{G}$ to be the filter generated by $\mathcal{G}_o$.

Maximal Filters

05° Let $\mathbf{F}$ be the family of all filters on X. Let us supply $\mathcal{F}$ with a partial ordering, as follows:

$$\mathcal{F}' \preceq \mathcal{F}' \quad \Longleftrightarrow \quad \mathcal{F}' \subseteq \mathcal{F}''$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are any filters on X. With respect to the partial ordering on $\mathbf{F}$ just defined, we plan to study the *maximal* filters. These are the filters $\mathcal{U}$ on X such that, for any filter $\mathcal{F}$ on X, if $\mathcal{U} \subseteq \mathcal{F}$ then $\mathcal{U} = \mathcal{F}$. Very often, one refers to such filters as *ultrafilters*.

06° Obviously, the principal filters on X are maximal with respect to the foregoing partial ordering. We inquire whether there are any others.

07° Let $\mathcal{U}$ be an ultrafilter on X. With reference to article 04°, we find that, for any subsets A and B of X, if $A \cup B \in \mathcal{U}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$. We infer that $\mathcal{U}$ meets the *finite union condition*, which is to say that, for any finite family $\mathcal{A}$ of subsets of X, if:

$$\bigcup \mathcal{A} \in \mathcal{U}$$

then there is at least one set A in $\mathcal{A}$ such that $A \in \mathcal{U}$.

08° In fact, the foregoing condition characterizes ultrafilters. To see that it is so, let us introduce a filter $\mathcal{F}$ on X which meets the finite union condition and let us suppose that $\mathcal{F}$ is not maximal. Accordingly, we may introduce a filter $\mathcal{G}$ on X and a subset A of X such that $\mathcal{F} \subseteq \mathcal{G}$, $A \notin \mathcal{F}$, and $A \in \mathcal{G}$. Now the subset A and its complement B in X yield $A \cup B \in \mathcal{F}$ while $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Consequently, the supposition is untenable. Hence, $\mathcal{F}$ is maximal.

09° By the foregoing discussion, we infer that, for any ultrafilter $\mathcal{U}$ on X, $\mathcal{U}$ is principal iff:

$$\bigcap \mathcal{U} \neq \emptyset$$

In fact, for any member ξ of X, if:

$$\xi \in \bigcap \mathcal{U}$$

then, for any V in $\mathcal{U}$, $\{\xi\} \cup (V \backslash \{\xi\}) \in \mathcal{U}$, hence, $\{\xi\} \in \mathcal{U}$, so that $\mathcal{U} = \mathcal{P}_\xi$.

Existence of Maximal Filters

10° From this point forward, let us assume that X is infinite.

11° Let $\mathcal{E}$ be the filter on X consisting of all subsets E for which the complement F of E in X is finite. In turn, let $\mathbf{F}_o$ be the family of all filters $\mathcal{F}$ on X such that $\mathcal{E} \subseteq \mathcal{F}$.

12• Verify that $\mathcal{E}$ is not maximal.

13° By a *chain* in $\mathbf{F}_o$, we mean a subfamily $\mathbf{C}$ of $\mathbf{F}_o$ such that, for any filters $\mathcal{F}'$ and $\mathcal{F}''$ in $\mathbf{C}$, $\mathcal{F}' \preceq \mathcal{F}''$ or $\mathcal{F}'' \preceq \mathcal{F}'$. We may say that $\mathbf{C}$ is *linearly* ordered. For such a family $\mathbf{C}$, we find that:

$$\mathcal{G} = \bigcup \mathbf{C}$$

is a filter in $\mathbf{F}_o$ and $\mathcal{G}$ is an upper bound for $\mathbf{C}$, in the sense that, for each filter $\mathcal{F}$ in $\mathbf{C}$, $\mathcal{F} \subseteq \mathcal{G}$.

14° By the foregoing observation, we conclude that every chain in $\mathbf{F}_o$ is bounded. Now the Lemma of Zorn implies that there exist filters $\mathcal{U}$ in $\mathbf{F}_o$ which are maximal. Obviously, such filters are maximal in $\mathbf{F}$ as well. And they are not principal.

The Space of Ultrafilters

15° Let X be any set. Let $\mathbf{U}$ be the family of all ultrafilters on X. For amusement, let us note that:

$$\mathbf{U} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$$

We intend to supply $\mathbf{U}$ with a topology. The corresponding topological space proves to have remarkable properties.

16° To that end, let A be any subset of X. Let $\mathbf{T}_A$ be the subset of $\mathbf{U}$ defined as follows:

$$\mathbf{T}_A = \{\mathcal{U} \in \mathbf{U} : A \in \mathcal{U}\}$$

These subsets of $\mathbf{U}$ form the *base* for the topology on $\mathbf{U}$, soon to be defined.

17• Note that $\mathbf{T}_\emptyset = \emptyset$ and $\mathbf{T}_X = \mathbf{U}$. Verify that:

$$B \subseteq C \Longrightarrow \mathbf{T}_B \subseteq \mathbf{T}_C$$

$$\mathbf{T}_{B\cap C} = \mathbf{T}_B \cap \mathbf{T}_C, \quad \mathbf{T}_{B\cup C} = \mathbf{T}_B \cup \mathbf{T}_C, \quad \mathbf{T}_{X\backslash D} = \mathbf{U}\backslash\mathbf{T}_D$$

where B, C, and D are any subsets of X.

18° In turn, let $\mathcal{A}$ be any subset of $\mathcal{P}(X)$. Let $\mathbf{T}_\mathcal{A}$ be the subset of $\mathbf{U}$ defined as follows:

$$\mathbf{T}_\mathcal{A} = \bigcup_{A\in\mathcal{A}} \mathbf{T}_A$$

These subsets of $\mathbf{U}$ form the topology on $\mathbf{U}$. They are the *open* subsets of $\mathbf{U}$. By definition, they are the various unions of families of basic open subsets of $\mathbf{U}$.

Properties

19° Now let us prove that the topological space $\mathbf{U}$ is hausdorff, compact, and extremely disconnected.

20° First, hausdorff. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be distinct ultrafilters in $\mathbf{U}$:

$$\mathcal{U}_1 \neq \mathcal{U}_2$$

Of course, there must be some subset A of X such that $A \in \mathcal{U}_1$ but $A \notin \mathcal{U}_2$. Hence, $B = X \backslash A \in \mathcal{U}_2$. Consequently:

$$\mathcal{U}_1 \in \mathbf{T}_A, \ \ \mathcal{U}_2 \in \mathbf{T}_B, \ \ \mathbf{T}_A \cap \mathbf{T}_B = \emptyset$$

It follows that $\mathbf{X}$ is hausdorff.

21° Second, compact. Let us introduce an open covering of $\mathbf{U}$:

$$\mathbf{C}_{\mathbf{A}} = \{\mathbf{T}_{\mathcal{A}} : \mathcal{A} \in \mathbf{A}\}$$

where $\mathbf{A}$ is a subset of $\mathcal{P}(\mathcal{P}(X))$. By the definition of covering:

$$\bigcup_{\mathcal{A} \in \mathbf{A}} \mathbf{T}_{\mathcal{A}} = \mathbf{U}$$

We must show that there is a finite subset $\mathbf{F}$ of $\mathbf{A}$ such that:

$$\bigcup_{\mathcal{A} \in \mathbf{F}} \mathbf{T}_{\mathcal{A}} = \mathbf{U}$$

To that end, let $\mathcal{B}$ be the subset of $\mathcal{P}(X)$ defined as follows:

$$\mathcal{B} = \bigcup \mathbf{A}$$

Obviously:

$$\bigcup_{B \in \mathcal{B}} \mathbf{T}_B = \bigcup_{\mathcal{A} \in \mathbf{A}} \Big(\bigcup_{B \in \mathcal{A}} \mathbf{T}_B \Big) = \bigcup_{\mathcal{A} \in \mathbf{A}} \mathbf{T}_{\mathcal{A}} = \mathbf{U}$$

Moreover, for any B in $\mathcal{B}$, there is some $\mathcal{A}$ in $\mathbf{A}$ such that $B \in \mathcal{A}$, so that:

$$\mathbf{T}_B \subseteq \mathbf{T}_{\mathcal{A}}$$

Now we need only show that there is a finite subset $\mathcal{F}$ of $\mathcal{B}$ such that:

$$(\circ) \qquad \bigcup_{B \in \mathcal{F}} \mathbf{T}_B = \mathbf{U}$$

In effect, we have reduced the context of a general covering of $\mathbf{U}$ by open subsets to the context of a basic covering of $\mathbf{U}$ by basic open subsets.

22° By the finite union condition, condition ($\circ$) is equivalent to the following condition:

$$(\bullet) \qquad \bigcup_{B \in \mathcal{F}} B = X$$

Let us suppose that there is no finite subset $\mathcal{F}$ of $\mathcal{B}$ such that condition ($\bullet$) holds true. It would follow that the family $\mathcal{C}$ of complements:

$$\mathcal{C} = \{C = X \backslash B : B \in \mathcal{B}\}$$

generates a filter on X. Consequently, there would be an ultrafilter $\mathcal{U}$ on X which includes $\mathcal{C}$. It would follow that:

$$\mathcal{U} \notin \bigcup_{B \in \mathcal{B}} \mathbf{T}_B$$

a contradiction. So the supposition is untenable. Hence, there is finite subset $\mathcal{F}$ of $\mathcal{B}$ such that condition ($\bullet$) holds true. The proof is complete.

23• Verify that, for any subset D of X, $\mathbf{T}_D$ is not only open but also compact.

24° Third, extremely disconnected. Let $\mathcal{A}$ be any subset of $\mathcal{P}(X)$. We will show that there is a subset B of X that:

$$(*) \qquad clo(\mathbf{T}_\mathcal{A}) = \mathbf{T}_B$$

In this way, we will prove that the closure of any open subset of $\mathbf{U}$ is itself open, in fact, that it is a basic open subset of $\mathbf{U}$.

25° To that end, let us introduce the following sets:

$$B = \bigcup \mathcal{A}, \quad C = X \backslash B, \quad \mathcal{B} = \mathcal{P}(B), \quad \mathcal{C} = \mathcal{P}(C)$$

One can easily check that $\mathbf{T}_\mathcal{C}$ is the largest (under the relation of inclusion) among all open subsets of $\mathbf{U}$ which are disjoint from $\mathbf{T}_\mathcal{A}$. Consequently:

$$\mathbf{U} \backslash \mathbf{T}_\mathcal{C} = clo(\mathbf{T}_\mathcal{A})$$

However, $\mathbf{T}_{\mathcal{B}} = \mathbf{T}_B$ and $\mathbf{T}_{\mathcal{C}} = \mathbf{T}_C$. It follows that:

$$\mathbf{T}_B = \mathbf{U} \backslash \mathbf{T}_C = clo(\mathbf{T}_{\mathcal{A}})$$

The proof is complete.

26$^\bullet$ For each member ξ of X, we may identify ξ with the corresponding principal ultrafilter $\mathcal{P}_\xi$. In this way, we obtain an injective mapping π carrying X to $\mathbf{U}$:

$$\pi(\xi) = \mathcal{P}_\xi$$

where ξ is any member of X. Show that the range of π is dense in $\mathbf{U}$:

$$clo(ran(\pi)) = \mathbf{U}$$

Show that, for each member ξ of X, $\pi(\xi)$ is an isolated point in $\mathbf{U}$. In fact:

$$\{\mathcal{P}(\xi)\} = \mathbf{T}_{\{\xi\}}$$