PERSPECTIVE DRAWING

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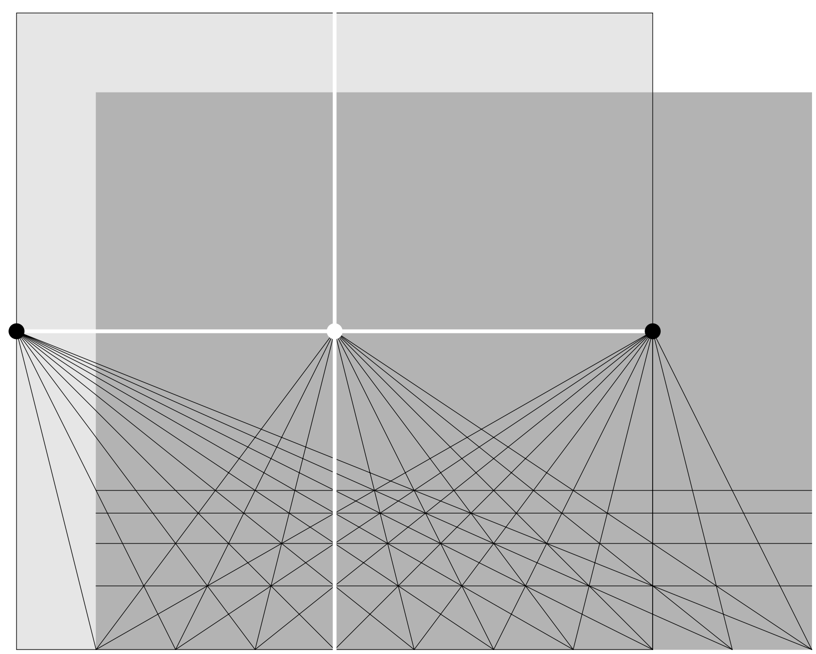
Introduction

01° We seek a method by which an artist may represent, faithfully, our solid world on a plane. In this adventure, we follow the lead of Leonardo da Vinci:

> "That is the most praiseworthy which is most like the thing represented."

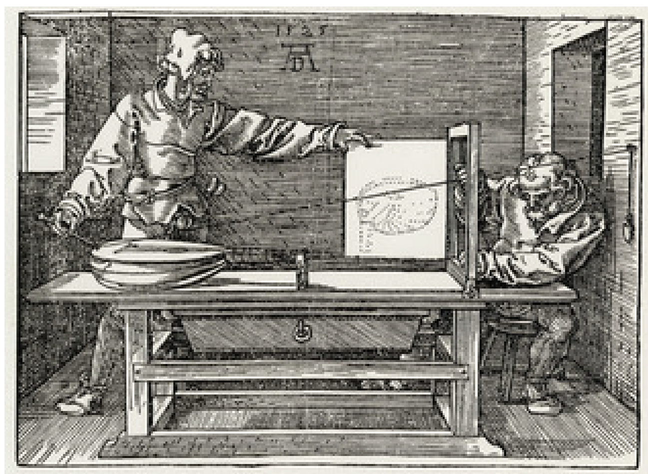
02° Actually, such a method was developed by the Greek geometer Euclid, three hundred years before the common era, in support of designs for stage settings in Greek plays. The method was redeveloped by Leon Battista Alberti (1435) and others in the early years of the Italian Renaissance. One may find a lucid discussion of the method in the third chapter of the book by Samuel Edgerton: The Renaissance Rediscovery of Linear Perspective. The method so described leads to the following template, a hybrid of the Vanishing Point (○) and the Focal Points (●).



Bifocal

03° However, from a logical point of view, the method stands undefended. The discussion by Edgerton (and many others) calls for confidence, taken for granted, in certain practical principles governing (single eye) visual perception. In this brief essay, we propose to supply a defense of the discussion, in terms of the far reaching analytic method of "coordinates" introduced by René Descartes (1637).

04° For orientation, one might enjoy inspecting the following woodcut by Albrecht Dürer, The Draughtsman (1524):



The Draughtsman

Coordinates

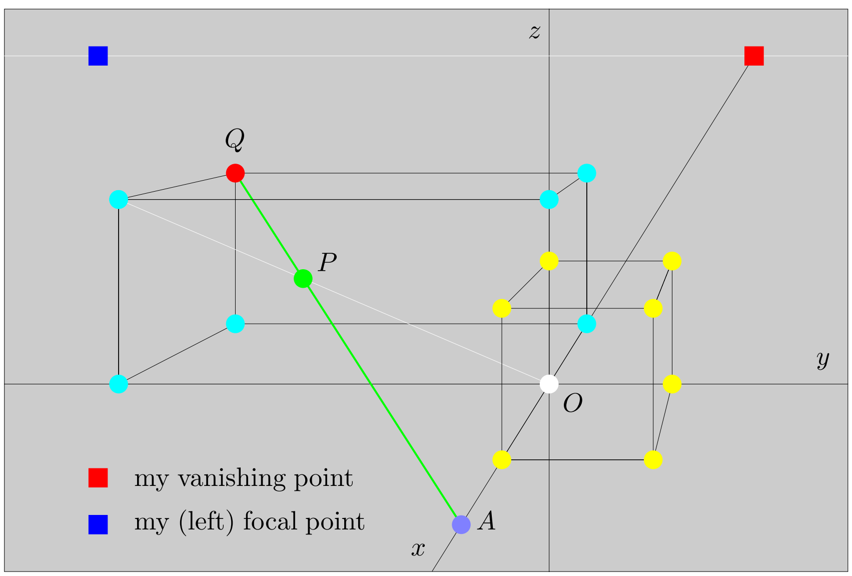
05° In 1637a, René Descartes published one of the celebrated works in the history of thought: **Discours de la Méthod**. In the preface to this work, Descartes set forth the central precepts of his scientific method, by which one would "acquire knowledge and avoid error." In the work itself, he presented three substantial discourses, which served as grand instances of successful application of his method. However, in course of time, the preface came to be viewed (and published) in its own right as a fundamental exposition of the scientific method.

06° The three discourses by Descartes were devoted to Geometry, Dioptrics, and Meteorology. In the first, Descartes initiated the study of geometry by arithmetic methods, that is, by means of coordinate systems. In the second, he described a quantitatively precise expression for the relation between the incident and refracted rays in context of refraction of light, the relation now known as the Law of Snell. Finally, in the third, he applied the Law of Snell to develop a compelling explanation of the Rainbow.

07° Just now, we will apply the first discourse, to exploit cartesian coordinates in planes and in space. We will consider the second and third discourses later.

Real $\Longrightarrow$ *Pictorial*

08° We may represent the basic problem of perspective drawing in terms of the following diagram. Actually, the diagram is itself a perspective drawing of the method of perspective drawing.



Analytic Method

09° We identify the plane of the canvas with the coordinate plane defined by the condition:

$$x = 0$$

We set the coordinates for the vanishing point O and the observer point A as follows:

$$O = (0, 0, 0), \quad A = (a, 0, 0)$$

10° One might leap to object that we have, without justification, placed the vanishing point at a preferred position. To counter the objection, we need only note that the origin of coordinates in the canvas plane can be set wherever we like. Once the vanishing point has been set, we may, without loss of generality, shift the origin of coordinates in the canvas plane to the more favorable position, occupied by the vanishing point.

11° Now we can state the basic problem of perspective drawing in sharp arithmetic terms. Given any point $Q = (x, y, z)$ in space, we must find the point $P = (0, v, w)$ on the canvas plane which lies on the line segment joining $A = (a, 0, 0)$ and $Q = (x, y, z)$. We contend that the solution stands as follows:

$$(\Pi) \qquad v = \frac{a}{a-x}\,y, \quad w = \frac{a}{a-x}\,z$$

12° To prove the contention, we argue as follows. The various points on the straight line joining A and Q stand in the form:

$$(1-t)A + tQ$$

where t is any number. That is:

$$((1-t)a, 0, 0) + (tx, ty, tz)$$

We seek the number t for which $(1-t)A + Q$ coincides with a point P on the canvas plane. That is:

$$((1-t)a + tx, ty, tz) = (0, v, w)$$

for suitable numbers v and w. It is the same to say that:

$$(1-t)a + tx = 0,\ ty = v,\ tz = w$$

Now, by simple algebra, we find that:

$$(a-x)t = a,\ v = \frac{a}{a-x}\,y,\ w = \frac{a}{a-x}\,z$$

We have proved the original contention (Π).

13° Now we can defend the practical principles governing (single eye) visual perception, by algebraic maneuvers. For instance, we claim that every straight line in the solid world must correspond to a straight line on the canvas plane.

14° To show that it is so, we introduce points Q_1 and Q_2 in the solid world and the corresponding points P_1 and P_2 on the canvas plane.:

$$Q_1 = (x_1, y_1, z_1),\ Q_2 = (x_2, y_2, z_2),\ P_1 = (0, v_1, w_1),\ P_2 = (0, v_2, w_2)$$

By (Π), we have:

$$v_1 = \frac{a}{a - x_1} y_1,\ \ w_1 = \frac{a}{a - x_1} z_1; \quad v_2 = \frac{a}{a - x_2} y_2,\ \ w_2 = \frac{a}{a - x_2} z_2$$

In turn, let Q be any point on the straight line defined by Q_1 and Q_2:

$$Q = (1 - t)Q_1 + tQ_2$$

where t is any number. Let P be the corresponding point on the canvas plane. We contend that P lies on the straight line (in the canvas plane) defined by P_1 and P_2:

$$P = (1 - s)P_1 + sP_2$$

for a suitable number s.

15° The coordinates of Q and P stand as follows:

$$Q = ((1 - t)x_1 + tx_2, (1 - t)y_1 + ty_2, (1 - t)z_1 + tz_2),\ \ P = (0, v, w),$$

related as required by (Π):

$$\begin{aligned} v &= \frac{a}{a - (1 - t)x_1 - tx_2}((1 - t)y_1 + ty_2)) \\ w &= \frac{a}{a - (1 - t)x_1 - tx_2}((1 - t)z_1 + tz_2)) \end{aligned}$$

It follows that:

$$s = t((1 - \beta t) + \beta)^{-1}$$

where:

$$\beta = (a - x_1)(a - x_2)^{-1}$$

16° Obviously, the correspondence between points on a straight line in the solid world and points on the corresponding straight line in the canvas plane is far from simple. But it must be so, since certain infinite straight lines in the solid world correspond, point for point, to finite straight lines in the canvas plane. See the following article.

17° We continue. Every straight line segment in the solid world which is perpendicular to the canvas plane will, under indefinite extension, seem to converge to the vanishing point O. We mean to say that, for fixed values of y and z:

$$x \longrightarrow -\infty \text{ implies } v \longrightarrow 0 \text{ and } w \longrightarrow 0$$

In this way, we defend the basic principle underlying perspective drawing.

18• We may identify the "ground plane" with the coordinate plane defined by the condition:

$$z = 0$$

Now every straight line segment in the solid world which is parallel to the ground plane and which makes an angle θ of $\pi/4$ radians with the canvas plane will, under indefinite extension, seem to converge to one of the focal points:

$$L = (0, -a, 0) \quad \text{or} \quad R = (0, +a, 0)$$

In fact, for a fixed value of z and for $x = \pm y$:

$$x \longrightarrow -\infty \text{ implies } v \longrightarrow \pm a \text{ and } w \longrightarrow 0$$

19° In general, every straight line segment in the solid world which is parallel to the ground plane and which makes an angle of θ radians with the canvas plane will, under indefinite extension, seem to converge to one of the points:

$$L = (0, -a\,tan(\theta), 0) \quad \text{or} \quad R = (0, +a\,tan(\theta), 0)$$

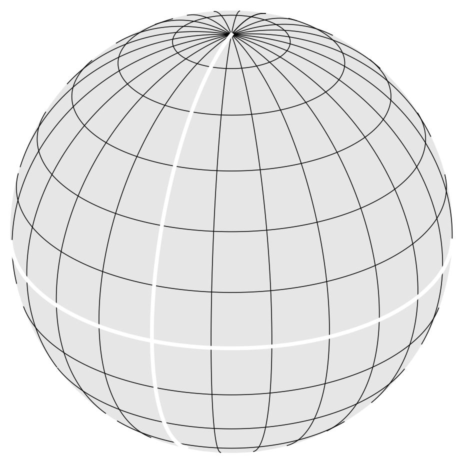
These are the various points on the centric line (that is, the horizontal line in the canvas plane passing through the vanishing point), defined by the conditions:

$$x = 0, \;\; z = 0$$

20° The foregoing technical observations are useful, in that they reveal metric information about the solid world implicit in an image, drawn in perspective, on the canvas plane. For instance, if a rectangle, let it be a table top, appears in the image on the canvas plane and if the orientation and the true proportions of the rectangle in the solid world are known (or can be presumed) then, from corresponding points of convergence on the centric line, one may deduce the eye point distance which figured in the drawing and, in turn, all relations of depth.

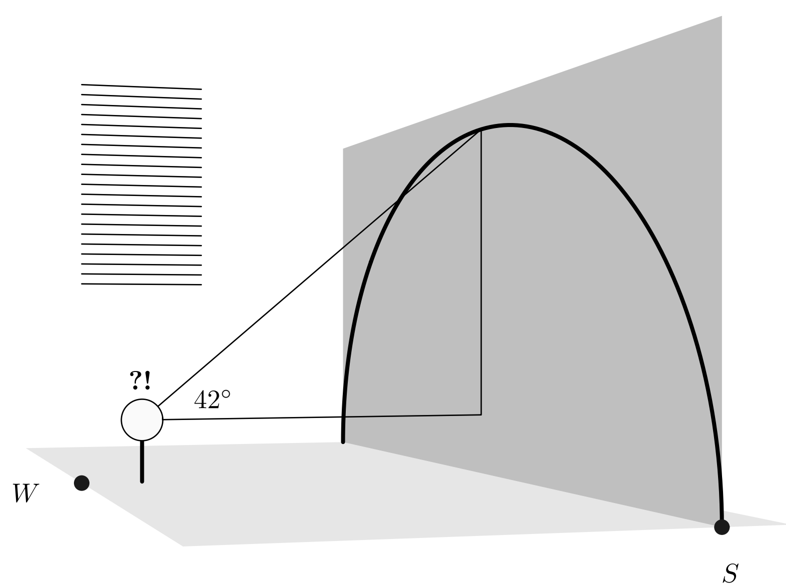
Examples

21°



The Sphere

22°



The Observer

23°

Saint Jerome

24° Dante (Convivio, 1304-1307)

“Geometry is most white in so far aa it is without stain or error, and is most certain in itself, and in its handmaiden, who is called perspective.”

25° Leonardo da Vinci (1497-1499)

“Among all the studies of natural causes and reasoning Light chiefly delights the beholder; and among the great features of mathematics the certainty of its demonstrations is what preeminently tends to elevate the mind of the investigator. Perspective, therefore, must be preferred to all the discourses and systems of human learning.”