The Optics of Euclid[1]

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DEFINITIONS

1. Let it be assumed that lines drawn directly from the eye pass through a space of great extent;

2. and that the form of the space included within our vision is a cone, with its apex in the eye and its base at the limits of our vision;

3. and that those things upon which the vision falls are seen, and that those things upon which the vision does not fall are not seen;

4. and that those things seen within a larger angle appear larger, and those seen within a smaller angle appear smaller, and those seen within equal angles appear to be of the same size;

5. and that things seen within the higher visual range appear higher, while those within the lower range appear lower;

6. and, similarly, that those seen within the visual range on the right appear on the right, while those within that on the left appear on the left;

7. but that things seen within several angles appear to be more clear.

Nothing that is seen is seen at once in its entirety. (Fig. 1.)

For let the thing seen be AD, and let the eye be B, from which let the rays of vision fall, BA, BG, BK, and BD. So, since the rays of vision, as they fall, diverge from one another, they could not fall in continuous line upon AD; so that there would be spaces also in AD upon which the rays of vision would not fall. So AD will not be seen in its entirety at the same time. But it seems to be seen all at once because the rays of vision shift rapidly.

Objects located nearby are seen more clearly than objects of equal size located at a distance. (Fig. 2.)

Let B represent the eye and let GD and KL represent the objects seen; and we must understand that they are equal and parallel, and let GD be nearer to the eye; and let the rays of vision fall, BG, BD, BK, and BL. For we could not say that the rays falling from the eye upon KL will pass through the points G and D. For in the triangle $BDLKGB$ the line KL would be longer than the line GD; but they are supposed to be of equal length. So GD is seen by more rays of the eye than KL. So GD will appear more clear than KL; for objects seen within more angles appear more clear.

Every object seen has a certain limit of distance, and when this is reached it is seen no longer. (Fig. 3.)

For let the eye be B, and let the object seen be GD. I say that GD, placed at a certain distance, will be seen no longer. For let GD lie midway in the divergence of the rays, at the limit of which is K. So, none of the rays from B will fall upon K. And the thing upon which rays do not fall is not seen. Therefore, every object seen has a certain limit of distance, and, when this is reached, the object is seen no longer.

Of equal spaces located upon the same straight line, those seen from a greater distance appear shorter. (Fig. 4.)

Let AB, BG, and GD represent equal spaces upon one straight line, and let the perpendicular AE be drawn, upon which let E represent the eye. I say that AB will appear longer than BG and BG longer than GD. For let the rays fall, EB, EG, and ED, and through the point B let BZ be drawn parallel to the straight line GE. Now AZ is equal to EZ. For, since parallel to GE, one side of the triangle AEG, the straight line BZ has been drawn, it follows also that EZ is related to ZA as GB to BA. So, as has been said AZ is equal to ZE. But the side BZ is longer than ZA; and so,

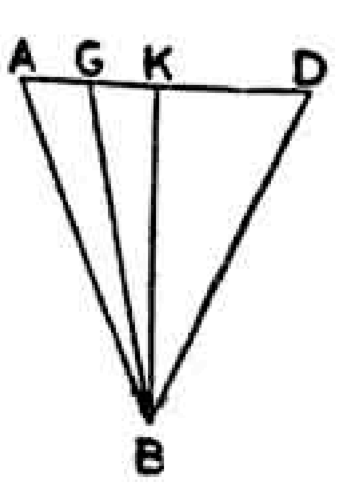


FIG. 1.

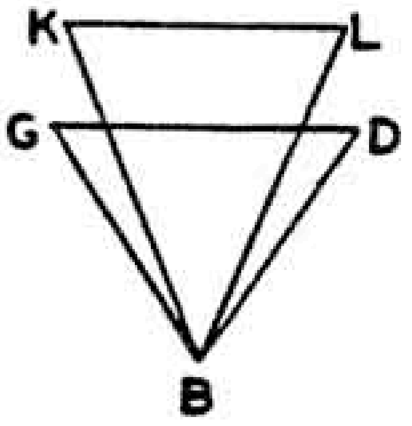


FIG. 2.

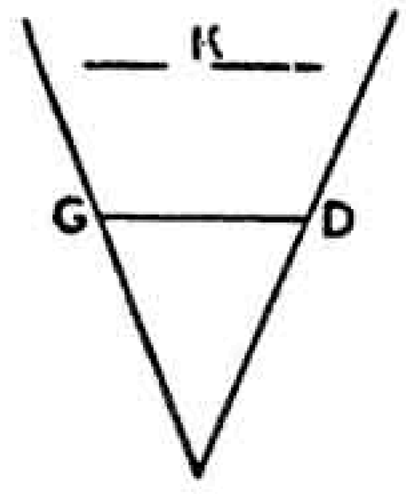


FIG. 3.

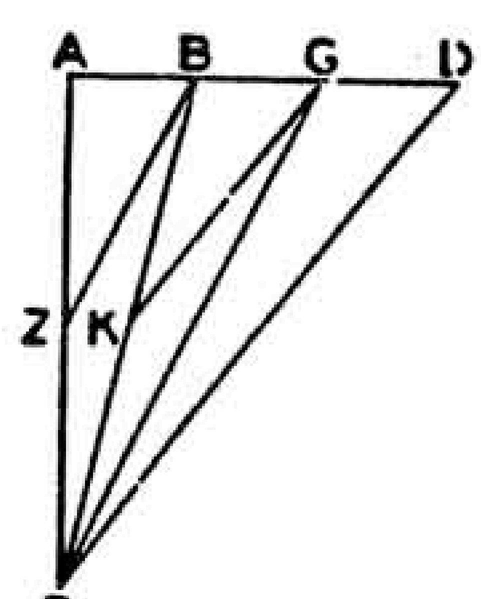


FIG. 4.

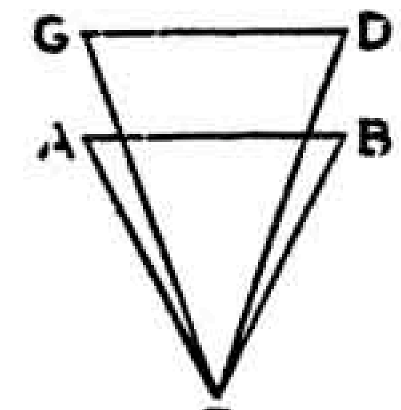


FIG. 5.

[1] Professor Charles N. Haskins, Professor of Mathematics at Dartmouth College, is largely responsible for this translation of the *Optics* of Euclid. A year ago, when he was doing research for the Dartmouth Eye Institute, he had occasion to use Euclid's essay and asked me if I would translate it. Strangely enough, it had never been translated into English. I agreed to undertake the task. Before the work was finished Professor Haskins died. The Dartmouth Eye Institute decided that the translation should be completed and published, and I wish to express my own gratitude to the Optical Society of America for its cooperation.

Euclid was a teacher of mathematics at Alexandria in the early part of the third century before Christ. Almost nothing is known of his life. He was a voluminous writer on mathematics and kindred subjects, his principal work being the Elements of Geometry in thirteen books. The *Optics* is an essay on the mathematics of optics. It is extant in two forms, one written by Euclid himself, the other a recension by Theon, written in the fourth century after Christ. In its original form, which is here translated, it is the earliest extant work on mathematical optics.

[Professor Burton died on March 20, 1945, before his translation of the *Optics* was set up in galley proof. His colleagues, in seeing the translation through the press, have endeavored only to secure in printed form the exact reproduction of Professor Burton's typewritten manuscript.]

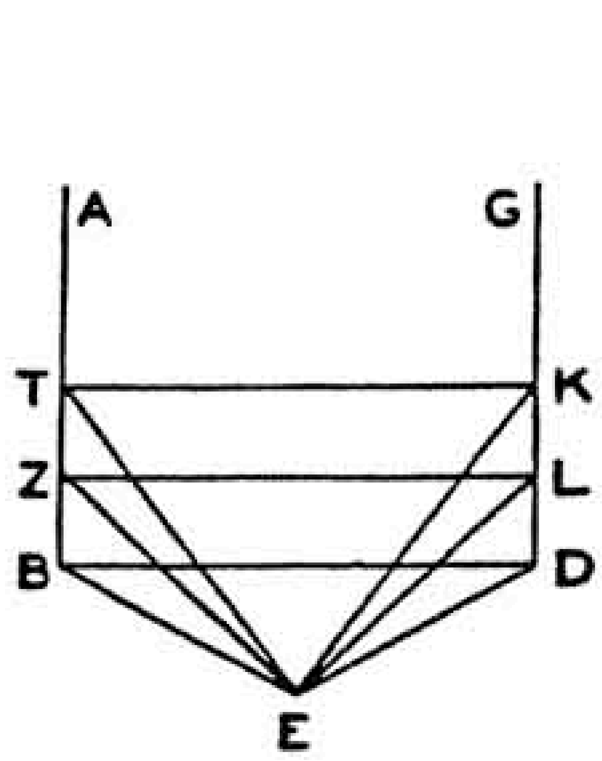


FIG. 6.

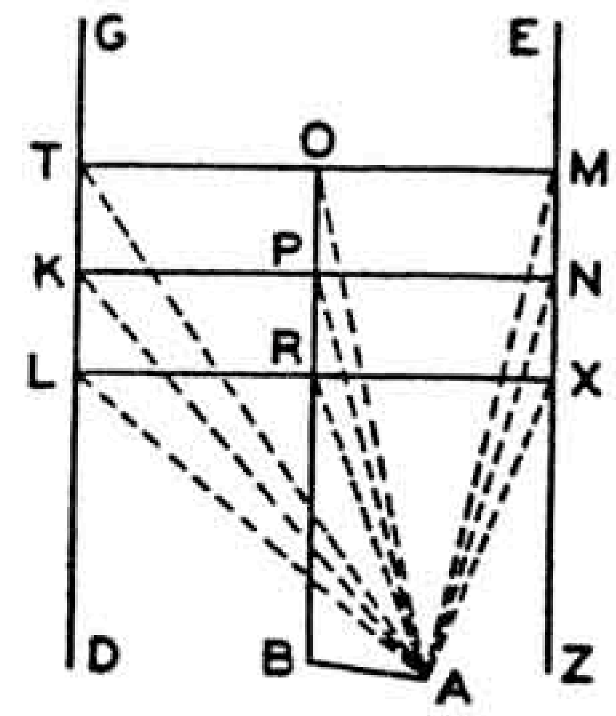


FIG. 6a.

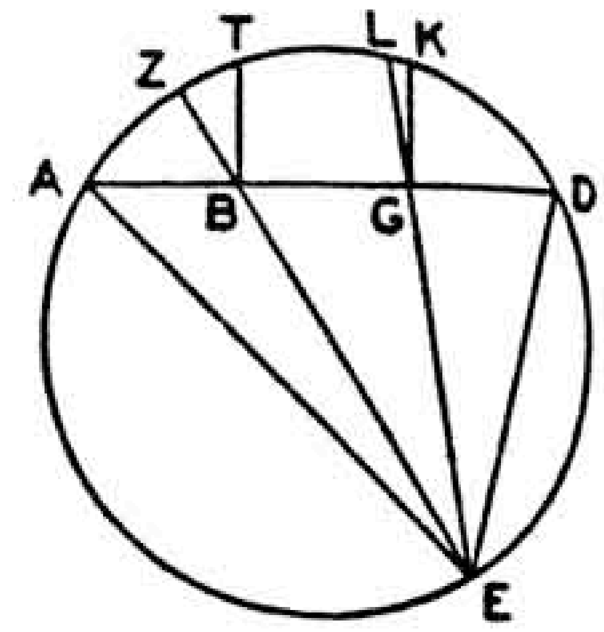


FIG. 7.

it is longer than ZE. Therefore the angle ZEB is greater than the angle ZBE; and the angle ZBE is equal to the angle BEG; and the angle ZEB is greater than the angle GEB. So, AB will seem longer than BG. Again in the same way, if through the point G a line parallel to DE is drawn, BG will seem longer than GD.

Objects of equal size unequally distant appear unequal and the one lying nearer to the eye always appears larger. (Fig. 5.)

Let there be two objects of equal size, AB and GD, and let the eye be indicated by E, from which let the objects be unequally distant, and let AB be nearer. I say that AB will appear larger. Let the rays fall, EA, EB, EG, and ED. Now, since things seen within greater angles appear larger, and the angle AEB is greater than the angle GED, AP will appear to be larger than GD.

Parallel lines, when seen from a distance, appear not to be equally distant from each other. (Fig. 6.)

Let there be two parallel lines, AB and GD, and let the eye be indicated by E. I say that AB and GD appear not to be parallel, and that the nearer space always appears greater than that farther away. Let the rays fall, EB, EZ, ET, ED, EL, and EK, and let these points be joined by the straight lines, BD, ZL, and TK. Now, since the angle BED is greater than the angle ZEL, BD appears longer than ZL. Again, since the angle ZEL is greater than the angle TEK, ZL appears longer than TK. So the space BD appears greater than the space ZL and the space ZL greater than the space TK. Therefore, lines equally distant from each other will no longer seem to be parallel, but will seem to be of varying distance apart.

In spaces lying at a higher level let the perpendicular AB be let down from the point A to the plane lying below, and let there be the parallels LX, KN, and TM. I say that so also the lines GD and EZ appear to be not parallel. (Fig. 6a.)

Let a perpendicular BR be drawn from B to LX, and let BR be continued to O, and let the rays fall, AL, AK, AT, AX, AN, and AM, and let AR, AP, and AO be connected. Now, since a straight line, AR, from a higher point has been connected with RX, then AR is a perpendicular upon RX, and AO upon OM, and AP upon PN. So, ARX, APN, and AOM are right-angled triangles. Since, then, they are right-angled triangles, and since PN is equal to RX, and since PA is longer than AR, the angle XAR is greater than the angle PAN. Therefore, RX will appear longer than PN. Similarly, RL appears longer than PK. So, the whole line LX will appear longer than the whole line KN. And thus the lines DG and ZE will seem to be a varying distance apart.

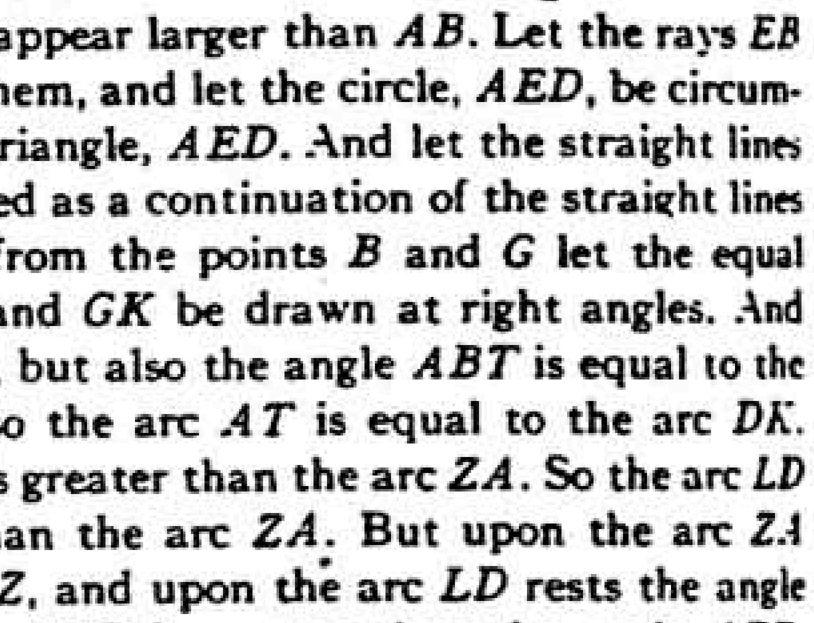
Objects of equal size upon the same straight line, if not placed next to each other and if unequally distant from the eye, appear unequal. (Fig. 7.)

Let there be two objects of equal size, AB and GD, upon the same straight line, but not next to each other, and unequally distant from the eye, E, and let the rays EA and ED fall upon them and let EA be longer than ED. I say that GD will appear larger than AB. Let the rays EB and EG fall upon them, and let the circle, AED, be circumscribed about the triangle, AED. And let the straight lines BZ and GL be added as a continuation of the straight lines EB and EG, and from the points B and G let the equal straight lines BT and GK be drawn at right angles. And AB is equal to GD, but also the angle ABT is equal to the angle DGK. And so the arc AT is equal to the arc DK. Thus, the arc KD is greater than the arc ZA. So the arc LD is much greater than the arc ZA. But upon the arc ZA rests the angle AEZ, and upon the arc LD rests the angle LED. So the angle LED is greater than the angle AEZ. But within the angle AEZ, AB is seen, and within the angle LED, GD is seen. Therefore, GD appears larger than AB.

Lines of equal length and parallel, if placed at unequal distances from the eye, are not seen in proportion to the distances. (Fig. 8.)

Let there be two lines, AB and GD, unequally distant from the eye E. I say that, as it appears, BE is not in the same relation to ED as GD is to AB. For let the rays fall, AE and EG, and with E as the central point and at the distance EZ let an arc be drawn, LZT. Now, since the triangle EZG is greater than the section EZL, and since

FIG. 8.



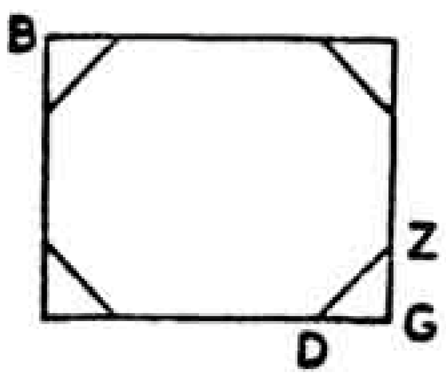


FIG. 9.

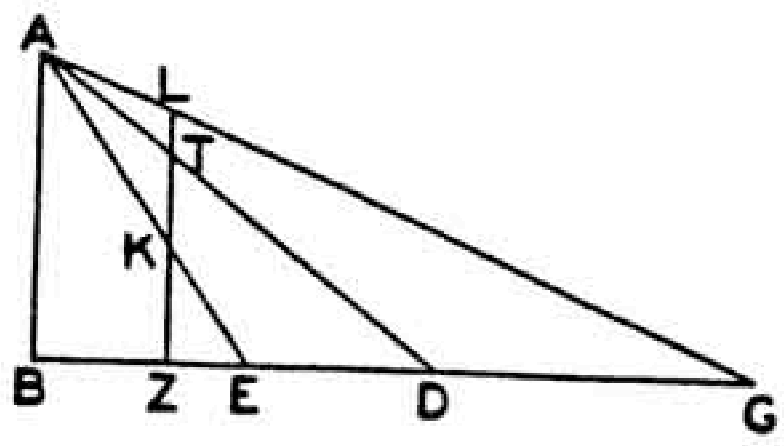


FIG. 10.

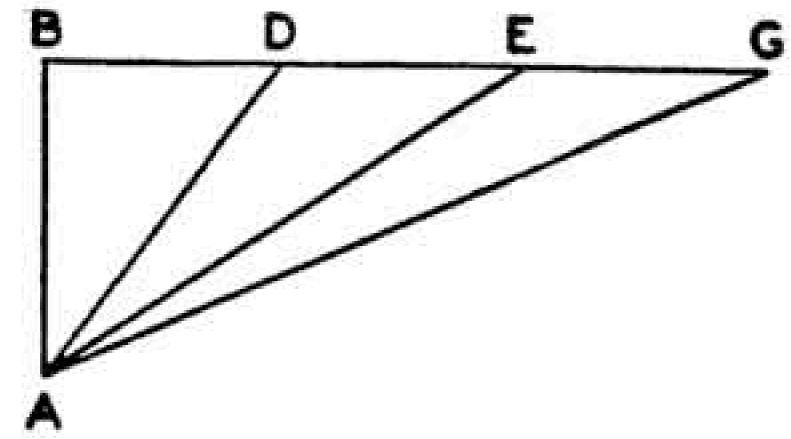


FIG. 11.

he triangle EZD is less than the section EZT, the triangle ZG, compared with the section EZL, has a greater ratio han the triangle EZD, when compared with the section ZT. And, alternately, the triangle EZG, compared with he triangle EZD, has a greater ratio than the section EZL as, when compared with the section EZT, and, when put ogether, the triangle EGD, compared with the triangle ZD, has a greater ratio than the section ELT, compared ith the section EZT. But as the triangle EDG is to the riangle EZD, so is GD to DZ. But GD is equal to AB, nd, as AB is to DZ, so is BE to ED. Now, BE in comarison with ED has a greater ratio than the section ELT n comparison with the section EZT. And as one section is o the other section, so the angle LET is to the angle ZET. o the straight line BE, compared with the straight line D, has a greater ratio than the angle LET compared with he angle ZET. And from the angle LET, GD is seen, and om the angle ZET, AB is seen. So lines of equal length re not seen in proportion to the distances.

Rectangular objects, when seen from a distance, appear und. (Fig. 9.)

For let the rectangle BG, standing upright, be seen from distance. Now, since each thing that is seen has a certain mit of distance beyond which it is no longer seen, the ngle G is not seen, but only the points D and Z are visible. n the same way also in the case of the other angles this ill happen. So that the whole thing will appear round.

In the case of flat surfaces lying below the level of the eye, he more remote parts appear higher. (Fig. 10.)

Let the eye be A, at a level higher than $BEDG$, and let he rays fall, AB, AE, AD, and AG, of which rays let AB e perpendicular upon the plane below. I say that GD ppears higher than DE, and DE higher than BE. For omewhere upon the line BE let the point Z be taken, and t the perpendicular ZL be drawn. Since the lines of ision fall upon ZL before they reach ZG, let the line AG eet ZL at the point L, the line AD at the point T, and he line AE at the point K. Now, since L is higher than T nd T higher than K, but G on the same line as L, and D on he same line as T, and E on the same line as K, and, since etween the lines AG and AD, DG is seen, and between the nes AD and AE, DE is seen, GD appears higher than DE. nd, similarly, DE will appear higher than BE; for things een by higher rays appear to be higher.

And it is clear that things seen on a higher plane will ppear to be concave.

In the case of flat surfaces located above the level of the eye, e parts farther away appear lower. (Fig. 11.)

Let the eye be indicated by A in a position lower than the plane BG, and let the rays fall, AB, AD, AE, and AG, and of these let AB be a perpendicular to the assumed plane. I say that GE appears lower than ED. Now, according to the theorem before stated, the ray AG is lower than AE, AE lower than AD, and AD lower than AB. But between the lines GA and AE, GE is seen, and between EA and AD, ED is seen, and between DA and BA, DB is seen. So GE appears lower than ED, and ED lower than DB.

In the case of lines extending forward, those on the right seem to be inclined toward the left, and those on the left seem to be inclined toward the right. (Fig. 12.)

Let two lines be seen, AB and GD, and let the eye be indicated by E, from which let the rays fall, ET, EK, EA, EZ, EL, and EG. I say that EZ, EL, and EG seem to be inclined toward the left, and ET, EK, and EA toward the right. For since EZ is more to the right than EL, and EL more than EG, hence EG seems to be inclined to the left of EL, and EL to the left of EZ. Similarly, also EA, EK, and ET seem to be inclined to the right.

In the case of lines of equal length, lying below the level of the eye, those lying farther away appear to be higher. (Fig. 13.)

Let there be lines of equal length, AB, GD, and EZ, and let the eye be indicated by L, located above these lines, and let the rays fall, LA, LG, and LE. I say that AB appears higher than GD, and GD higher than EZ. For, since LA is higher than LG, and LG higher than LE, and since the points A, G, and E are on the lines LA, LG, and LE, and since the lines AB, GD, and EZ extend from the points A, G, and E, then AB seems higher than GD, and GD higher than EZ.

In the case of lines of equal length lying above the level of the eye, those farther away appear lower. (Fig. 14.)

Let there be lines of equal length, AB, GD, and EZ, lying above the level of the eye, L. I say that AB appears lower than GD, and GD lower than EZ. Let the rays fall, LB, LD, and LZ. So, since the ray LB is lower than LD, and LD lower than LZ, but the points B, D, and Z are on the lines LB, LD, and LZ, and since the lines AB, GD, and EZ extend from B, D, and Z, then AB appears lower than GD, and GD lower than EZ.

In the case of objects below the level of the eye which rise one above another, as the eye approaches the objects, the taller one appears to gain height, but as the eye recedes, the shorter one appears to gain. (Fig. 15.)

Let there be two unequal lines, AB and GD, and let AB be the taller, and let E be the eye, and from this let a ray,

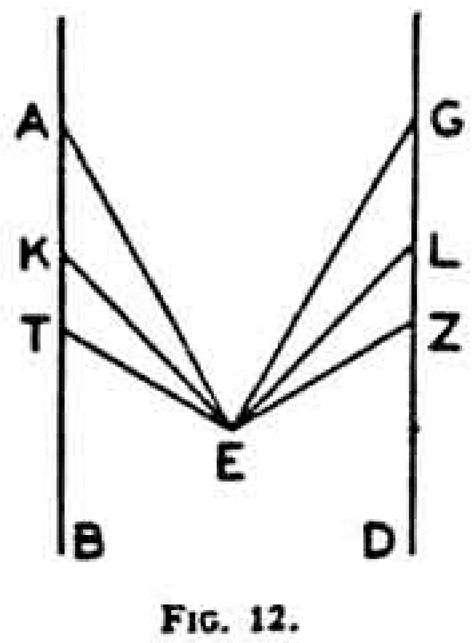


FIG. 12.

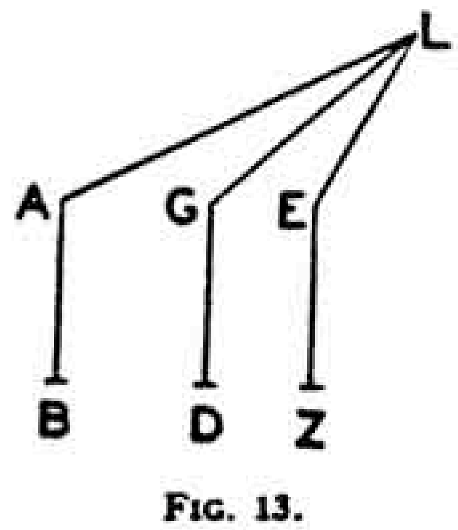


FIG. 13.

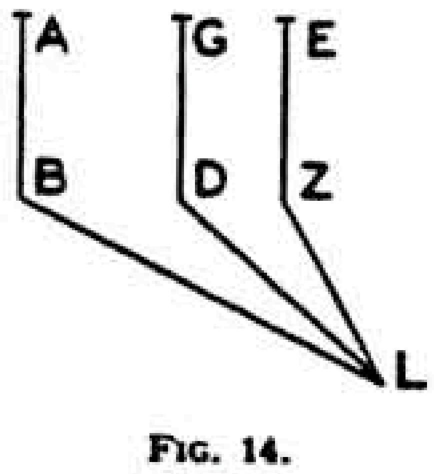


FIG. 14.

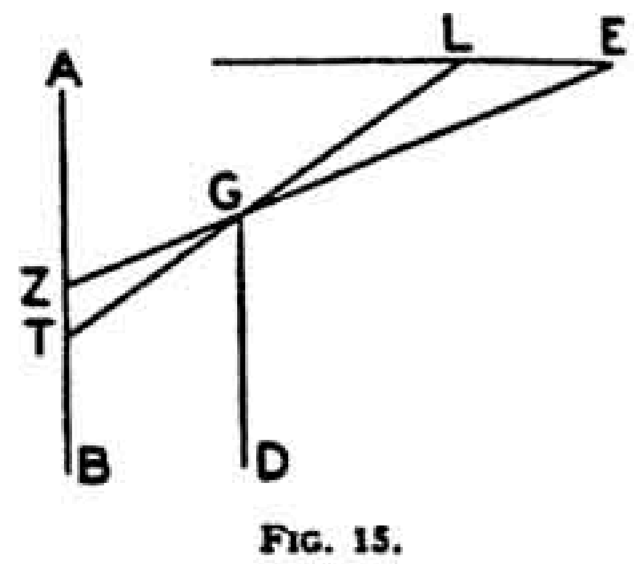


FIG. 15.

EZ, fall through *G*. Now, since below the level of the eye and below the ray *EZ*, *ZB*, and *GD* are seen, *AB* appears above *GD* by the length of *AZ*. Let the eye be moved nearer and let it be *L*, and from this let the ray *LT* fall through *G*. Then, since below the level of the eye and below the ray *LT*, *GD*, and *TB* are seen, *AB* will appear taller than *GD* by the length of *AT*. And from the point *E*, *AB* was seen to be taller by the length of *AZ*, and *AT* is longer than *AZ*. So, as the eye approaches the objects, the taller one appears to gain height, but, as the eye recedes, the shorter one appears to gain.

In the case of objects of unequal size above the level of the eye which rise one above another, as the eye approaches the objects, the shorter one appears to gain height, but, as the eye recedes, the taller one appears to gain. (Fig. 16.)

Let there be the lines *AB* and *GD*, of unequal length, and of these let *AB* be the taller. Let *E* represent the eye, and from this let the ray *EZ* fall through *G*. Now, since the lines *ZB* and *GD* are included under the ray *EZ*, *BZ* and *GD* appear equal to each other. So, *AB* appears taller than *GD* by the length of *AZ*. Now let the eye be moved nearer, and let it be *L*, from which let the ray *LT* fall through *G*. Now, since *BT* and *GD* are included under the ray *LT*, and since *ZB* and *GD* are included under the ray *EZ*, and since *ZA* is longer than *AT*, as the eye approaches the objects the shorter one appears to gain height, but, as the eye recedes, the taller one appears to gain.

In the case of objects that rise one above another, if the eye approaches the lesser one or recedes from it at a right angle, the taller one will always appear to rise above the shorter one by an equal amount. (Fig. 17.)

Let there be two lines of unequal length, *AB* and *GD*, and let *AB* be the longer, and let the eye be *Z*, located on a line at right angles to the end of *GD*, that is, at *G*. I say that whether the eye *Z* approaches or recedes, if it is on the same horizontal line, *AB* will seem to be taller than *GD* by an equal amount. For, let the ray *ZE* fall through *G*. Thus, *AB* appears taller than *GD* by the length of *AE*. Now let the eye be moved, and let it be farther away, and let it be *L* on the horizontal line. So, the ray, falling from the eye *L*, will pass through the point *G* and will be continued as far as the point *E*, and *AB* will rise above *GD* by the same amount.

To know how great is a given elevation when the sun is shining. (Fig. 18.)

Let the given elevation be *AB*, and we have to know how great it is. Let the eye be *D*, and let *GA* be a ray of the sun falling upon the end of the line *AB*, and let it be prolonged as far as the eye *D*. And let *DB* be the shadow of *AB*. And let there be a second line, *EZ*, meeting the ray, but not at all illuminated by it below the end of the line at *Z*. So, into the triangle *ABD* has been fitted a second triangle, *EZD*. Thus, as *DE* is to *ZE*, so is *DB* to *BA*. But the ratio of *DE* to *EZ* is known; and, therefore, the ratio of *DB* to *BA* is known. Moreover, *DB* is known; so, *AB* also is known.

To know how great is a given elevation when the sun is not shining. (Fig. 19.)

Let there be a certain elevation, *AB*, and let the eye be *G*, and let it be necessary to know how high is *AB* when the sun is not shining. Let a mirror be placed, *DZ*, and let *DB* be prolonged in a straight line continuous with *ED*, to the point where it touches *B*, the end of *AB*, and let a ray fall, *GL*, from the eye *G*, and let it be reflected to the point where it touches *A*, the end of *AB*, and let *ET* be a prolongation of *DE*, and from *G* let the perpendicular *GT* be drawn upon *ET*. Now, since the ray *GL* has fallen and the ray *LA* has been reflected, they have been reflected at equal angles, as is said in the *Catoptrica;* thus, the angle *GLT* is equal to the angle *ALB*. But also the angle *ABL* is equal to the angle *GTL*; and the remaining angle *LGT* is equal to the remaining angle *LAB*. So, the triangle *ALB* is

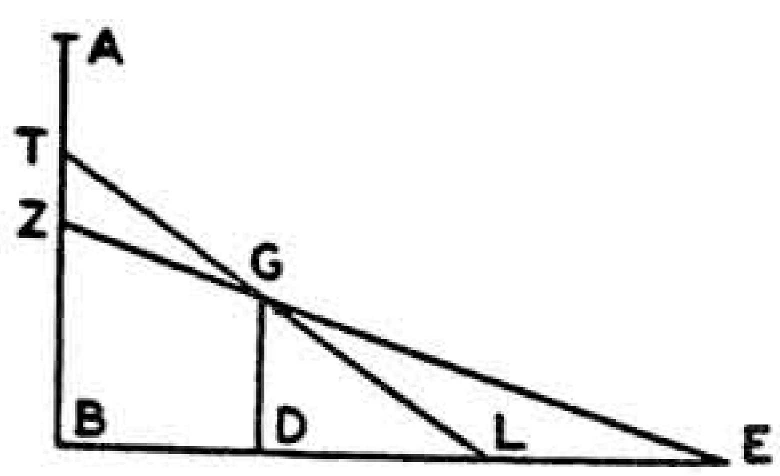


FIG. 16.

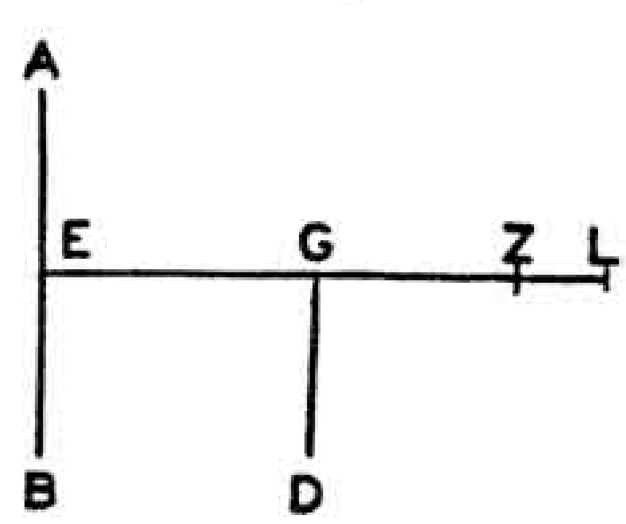


FIG. 17.

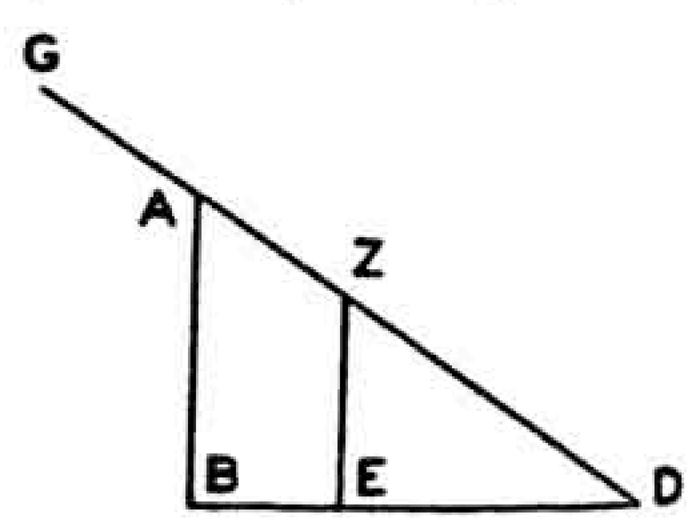


FIG. 18.

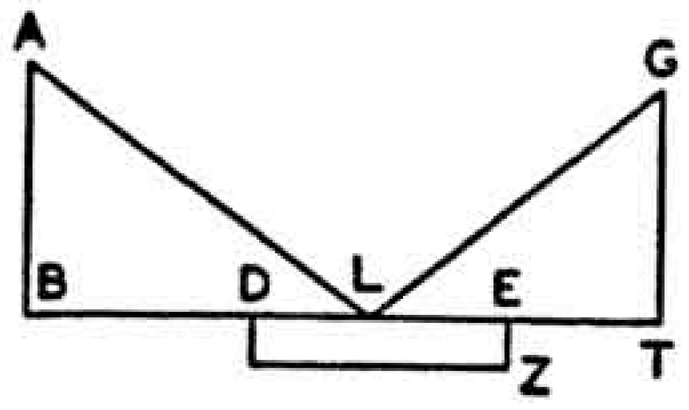


FIG. 19.

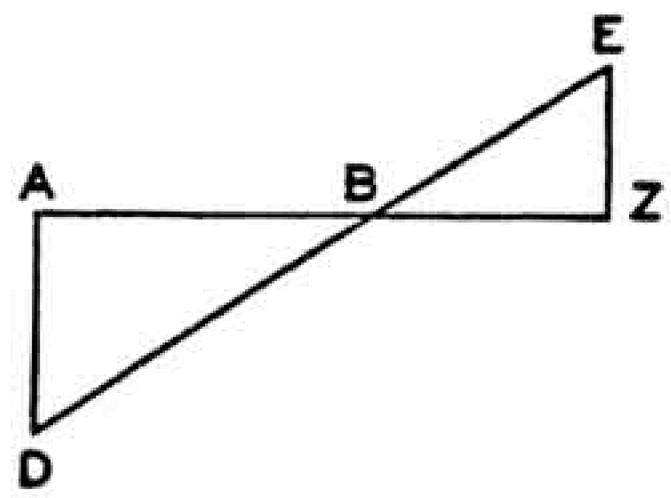


FIG. 20.

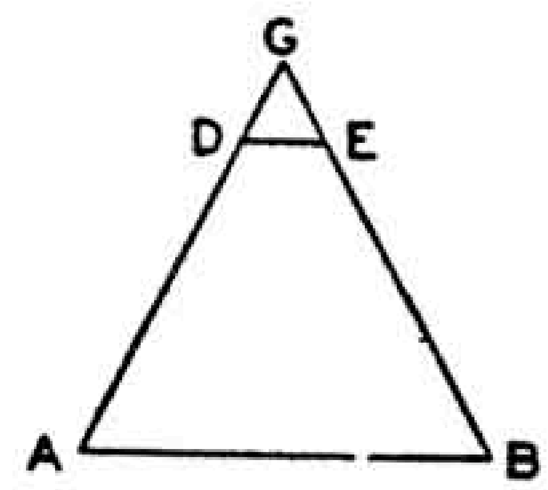


FIG. 21.

equiangular with the triangle GLT. But the sides of equiangular triangles are proportionate. Thus, as GT is to TL, so is AB to BL. But the ratio of GT to TL is known; and the ratio of BA to BL is known. But LB is known. And so, AB is known.

To know how great is a given depth. (Fig. 20.)

Let the given depth be AD, and let the eye be E, and let it be necessary to know how great is the depth. Let a ray of the sun fall before the eye, meeting the plane at the point B, and the depth at the point D. And let BZ be continued from B in a straight line, and let the perpendicular EZ be drawn from E to the horizontal line BZ. Now, since the angle EZB is equal to the angle BAD and the angle ABD is equal to the angle EBZ, the third angle also, BEZ, is equal to the angle ADB. So, the triangle ADB is equiangular with the triangle BEZ. And thus the sides will be proportionate. Then, as EZ is to ZB, so is DA to AB. But the ratio of EZ to ZB is known; and the ratio of DA to AB is known. And also AB is known. And thus AD is known.

To know how great is a given length. (Fig. 21.)

Let the given length be AB, and let the eye be G, and let it be necessary to know how great is the length of AB. Let the rays fall, GA and GB, and near the eye, G, let the point D be taken somewhere upon the ray (GA), and through the point D let the horizontal line DE be drawn, parallel to AB. Now, since DE has been drawn parallel to BA, one of the sides of the triangle ABG, as GD is to DE, so is GA to AB. But the ratio of GD to DE is known; and the ratio of AG to AB is known. And AG is known. So, AB also is known.

If an arc of a circle is placed on the same plane as the eye, the arc appears to be a straight line. (Fig. 22.)

Let BG be an arc lying on the same plane as the eye A, from which let the rays fall, AB, AD, AE, AZ, AL, AT, and AG. I say that the arc BG appears to be a straight line. Let K be the center of the arc, and let the straight lines KB, KD, KE, KZ, KL, KT, and KG be drawn. Now, since KB is seen within the angle KAB, and KD within the angle KAD, KB will appear longer than KD, and KD longer than KE, and KE longer than KZ, and, on the other side, KG will appear longer than KT, and KT longer than KL, and KL longer than KZ. On this account, KA remaining a straight line, BG is always a perpendicular. And the same things will happen also in the case of a concave arc.

An addition. It is possible to say these things also with reference to the visual rays themselves, that the ray between the eye A and the diameter is shortest, and that the ray nearer to it is always shorter than the one farther away. And the same things happen when AZ is a perpendicular upon the diameter. On this account the arc gives an impression of a straight line, and especially if it should be seen from a greater distance, so that we do not perceive the curve. On this account ropes not stretched tightly, when seen from the side, appear to sag, but seen from below, seem to be straight, and also the shadows of rings lying on the same plane with the source of light become straight.

An addition. If an arc of a circle is placed on the same plane as the eye, the arc appears to be a straight line. (Fig. 22a.)

Let BG be an arc, and let D be the eye, on the same plane as the arc BG, from which let the rays fall, DB, DZ, and DG. So, since nothing that is seen is seen at once in its entirety, BZ is a straight line. Similarly, also ZG. Thus, the whole arc, BG, will seem straight.

Of a sphere seen in whatever way by one eye, less than a hemisphere is always seen, and the part of the sphere that is seen itself appears as an arc. (Fig. 23.)

Let there be a sphere, of which A is the center, and let B be the eye. And let A and B be joined, and let the plane be continued along the line BA. So it will make a circular section. Let it make the circle $GDTL$, and around the diameter AB let the circle GBD be inscribed, and let the straight lines be drawn, GB, BD, AD, and AG. Now, since AGB is a semicircle, the angle AGB is a right angle; similarly, also the angle BDA. So, the lines GB and BD

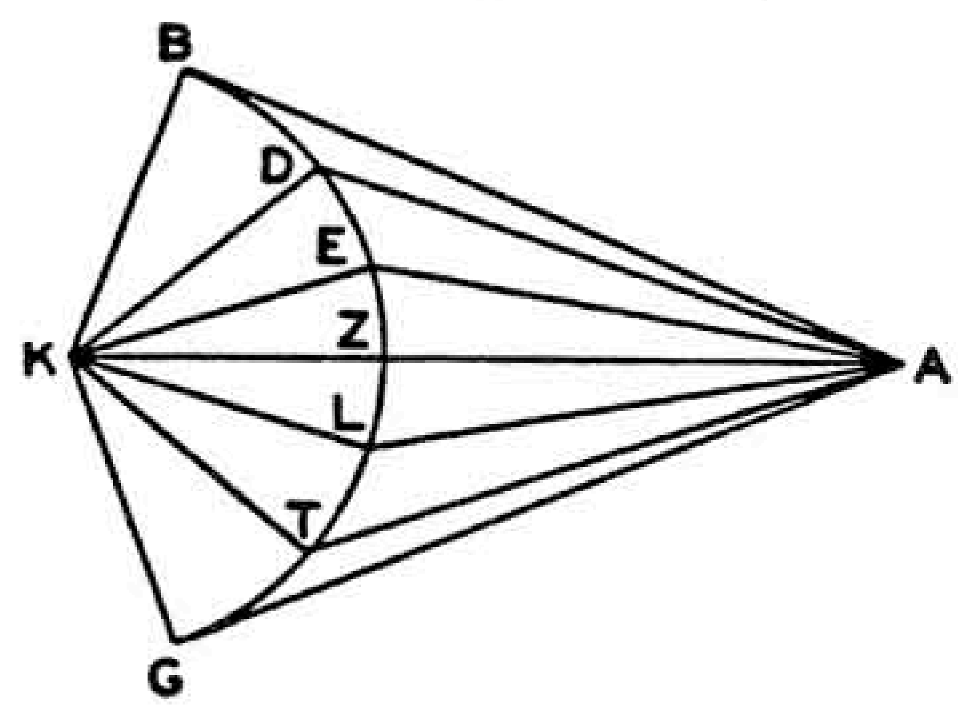


FIG. 22.

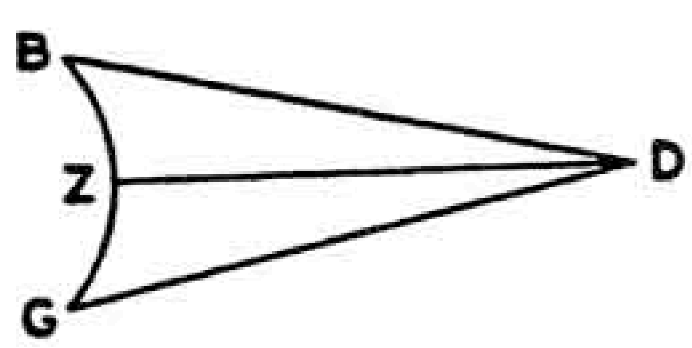


FIG. 22a

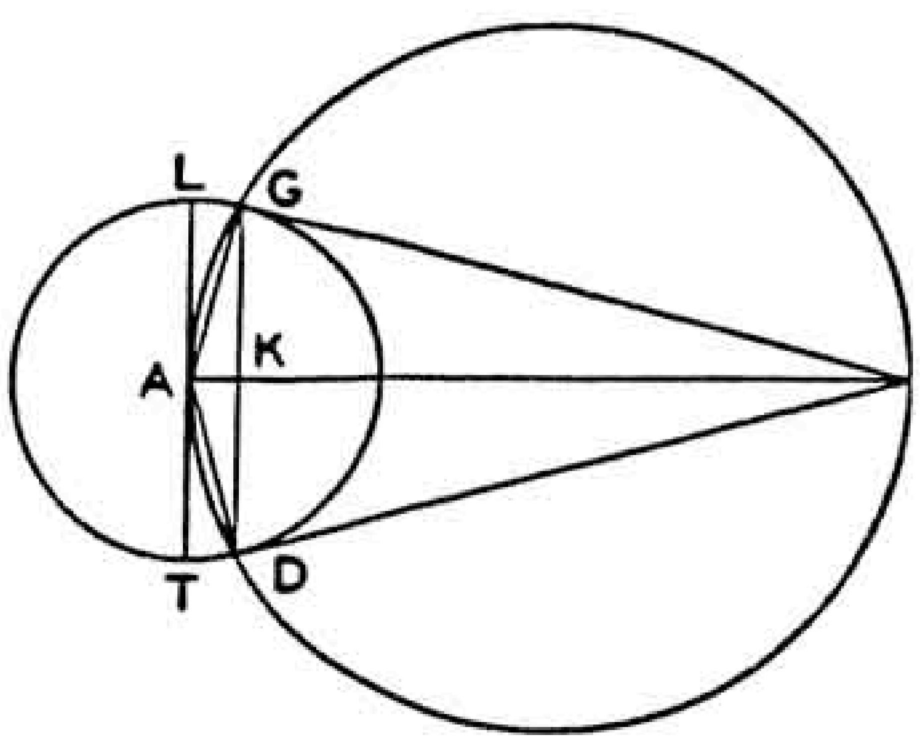


FIG. 23.

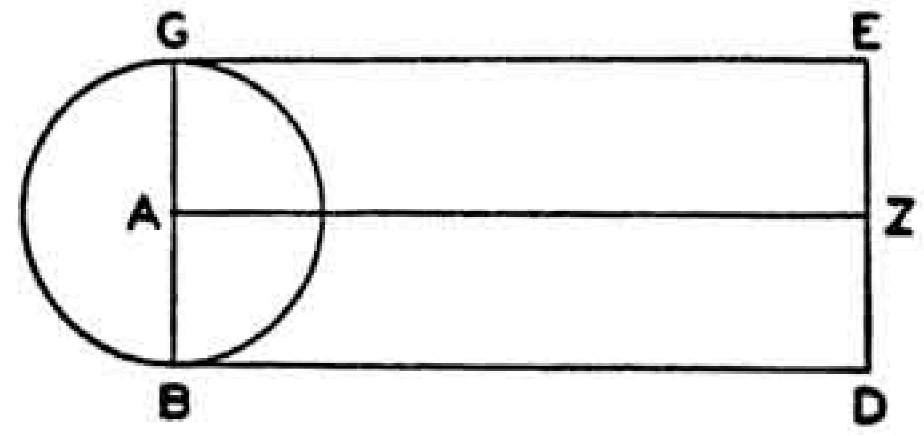


FIG. 25.

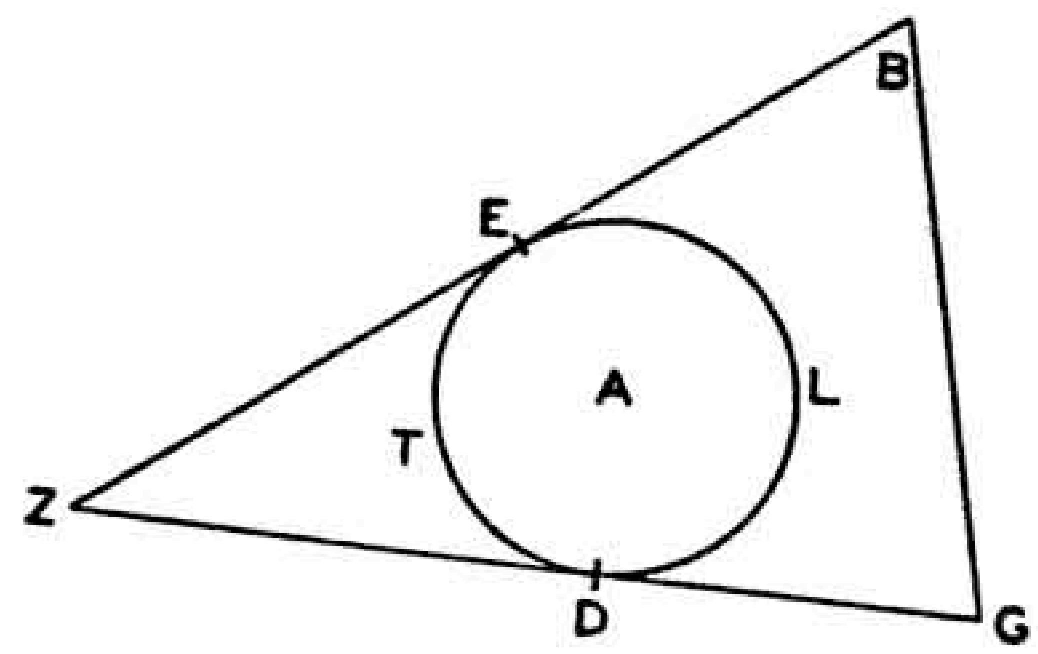


FIG. 26.

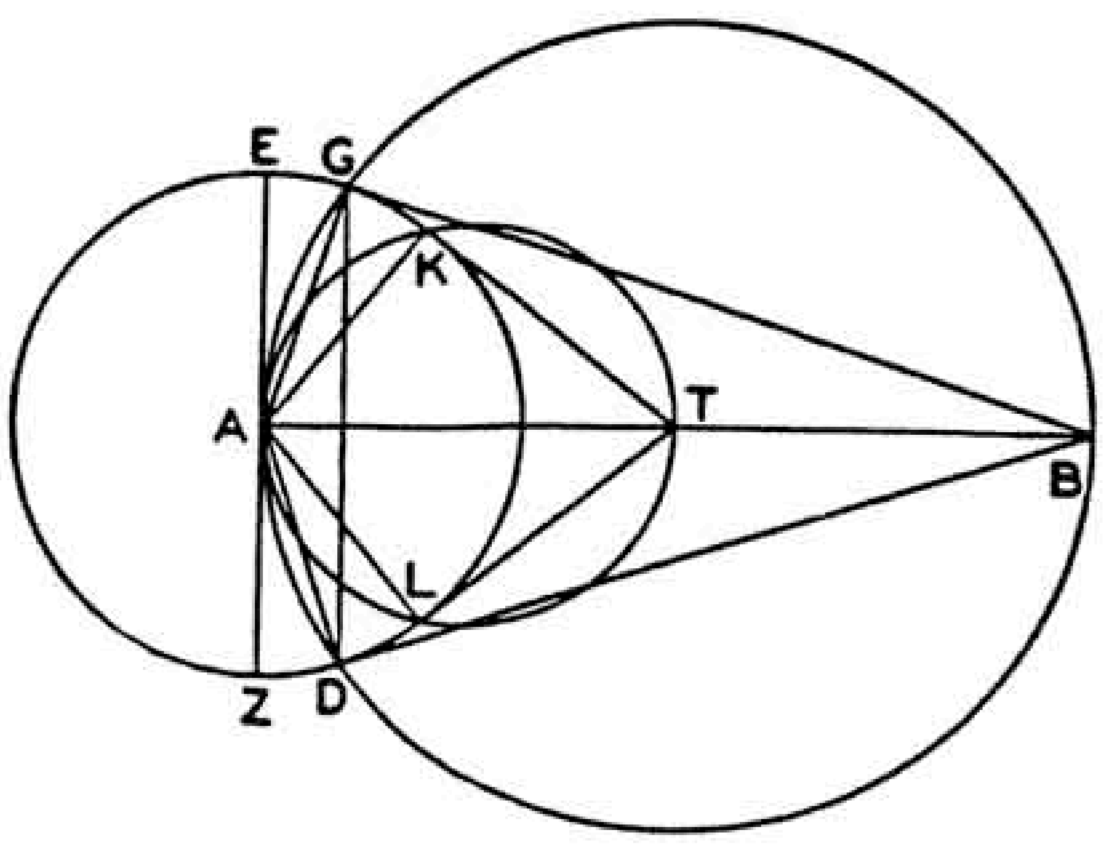


FIG. 24.

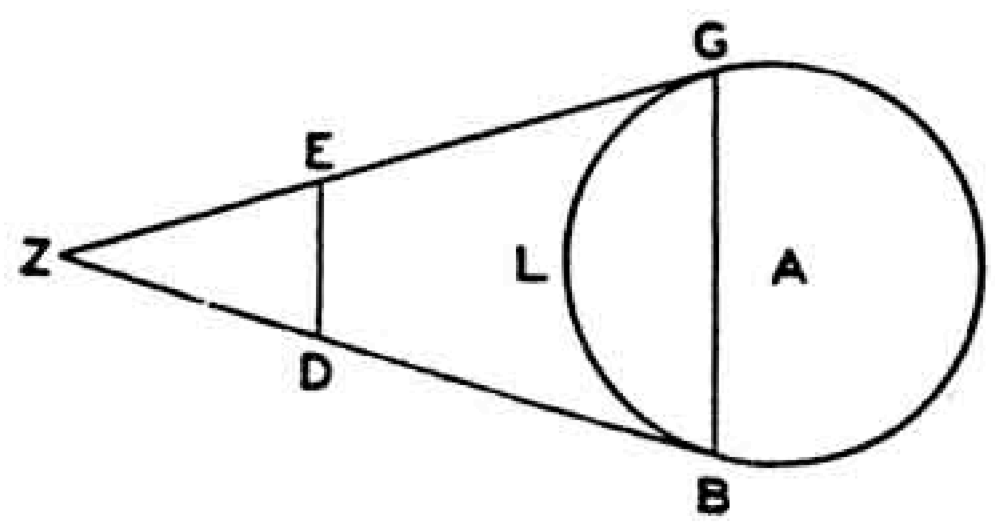


FIG. 27.

touch the sphere. Now, let G and D be joined, and let LT be drawn through the point A parallel to GD. So, the angles at K are right angles. Now, with AB remaining in its place, if the triangle BGK is revolved about the right angle K, and is restored to the same position from which it started, the line BG will touch the sphere at one point and the line KG will make a circular section. So, an arc will be seen in the sphere. And I say that it is less than a hemisphere. For, since LT is a semicircle, GD is less than a semicircle. And the same part of the sphere is seen by the rays BG and BD. So, GD is less than a hemisphere; and it is seen by the rays BG and BD.

When the eye approaches the sphere, the part seen will be less, but will seem to be more. (Fig. 24.)

Let there be a sphere, of which the center is A, and let the eye be B, from which let the straight line AB be drawn. And around AB let the circle GBD be inscribed, and from the point A let the straight line EZ be drawn, perpendicular to the straight line AB in either direction, and let the plane be produced along EZ and AB. So it will make a circular section. Let it be $GEZD$, and let GA, AD, DB, BG, and GD be drawn. So, according to the theorem given before, the angles at the points G and D are right angles. Thus, BG and BD, whatever rays there are, touch the sphere. And the part of the sphere, GD, is seen by the eye, B. Now let the eye be moved nearer to the sphere, and let it be at T, from which let the straight line TA be drawn, and let the circle ALK be inscribed, and let the straight lines TK, KA, AL, and LT be drawn. Now, similarly, by the eye at T is seen the part of the sphere, KL, and by the eye at B the part GD was seen. And KL is less than GD. So, as the eye approaches, the part seen is less. But it seems to be more; for the angle KTL is greater than the angle GBD.

When a sphere is seen by both eyes, if the diameter of the sphere is equal to the straight line marking the distance of the eyes from each other, the whole hemisphere will be seen. (Fig. 25.)

Let there be a sphere, of which A is the center, and on the sphere let the circle BG be inscribed about the center A, and let BG be drawn as its diameter, and at right angles from B and G let lines be drawn, BD and GE, and let DE be parallel to BG, and upon this (DE), let D and E represent the eyes. I say that the complete hemisphere will be seen. Through A let AZ be drawn parallel to each of the lines, BD and GE; then $ABZD$ is a parallelogram. Now, if the inscribed figure is revolved and then restored to the same position whence it started, AZ remaining in its place, it will start from B and B will come over G, and the figure formed under AB will be a circle through the center of the sphere. So a hemisphere will be seen by the eyes D and E.

If the distance between the eyes is greater than the diameter of the sphere, more than the hemisphere will be seen. (Fig. 26.)

Let there be a sphere, of which A is the center, and about the center A let the circle $ETDL$ be inscribed, and let the eyes be B and G, and let the distance between the eyes and G be greater than the diameter of the sphere, and let and G be joined. I say that more than the hemisphere will e seen. Let the rays fall, BE and GD, and let them be roduced beyond the points E and D; they approach each ther because the diameter is less than BG. So let them ieet at the point Z. Now, since from some point outside he circle the straight lines ZE and ZD have touched the ircumference, DTE is less than a semicircle. So, ELD is iore than a semicircle. But ELD is seen by the eyes at l and G. So, more than half of the circle will be seen by he eyes at B and G. The same part of the sphere too will e seen.

If the distance between the eyes is less than the diameter if the sphere, less than a hemisphere will be seen. (Fig. 27.)

Let there be a sphere, the center of which is the mark A, ind let the circle BG be inscribed around the mark A, ind let the distance between the eyes, DE, be less than the liameter of the sphere, and from the eyes let the same rays e drawn, DB and EG, touching the sphere. I say that less han a hemisphere will be seen. For let BD and GE be produced; they will fall upon the section GLB, since DE is less than the diameter of the sphere. Let them meet at the point Z. Now, since from a certain mark, Z, the straight lines have fallen, ZG and ZB, BLG is less than a semicircle. But the section of the sphere corresponds to the section BLG. So, the rays include less than a hemisphere.

If a cylinder in whatever way is seen by one eye, less than half a cylinder will be seen. (Fig. 28.)

Let there be a cylinder, and let the mark A be the center of its base, and let the circle BG be inscribed around A, and let the eye D be on the same plane as BG, the base of the cylinder, and from D to A let the line DA be drawn, and from D let the rays DB and DG be drawn, and let them touch the cylinder, and from the marks B and G let the sides of the cylinder rise at right angles, and let the plane through DB and BE and the one through DG and GZ be produced. So, neither of them cuts the cylinder; for DB, DG and BE, GZ touch the cylinder. Therefore, BG is seen by the rays BD and DG, and BG is less than a semicircle. In the same way less than half a cylinder will be seen.

If a cylinder should be seen by two eyes, it is clear that what has been said regarding the sphere will be true also in this case.

An addition. (Fig. 28a.) Let there be a circle, and let A be its center, and outside let there be the point Z, and from

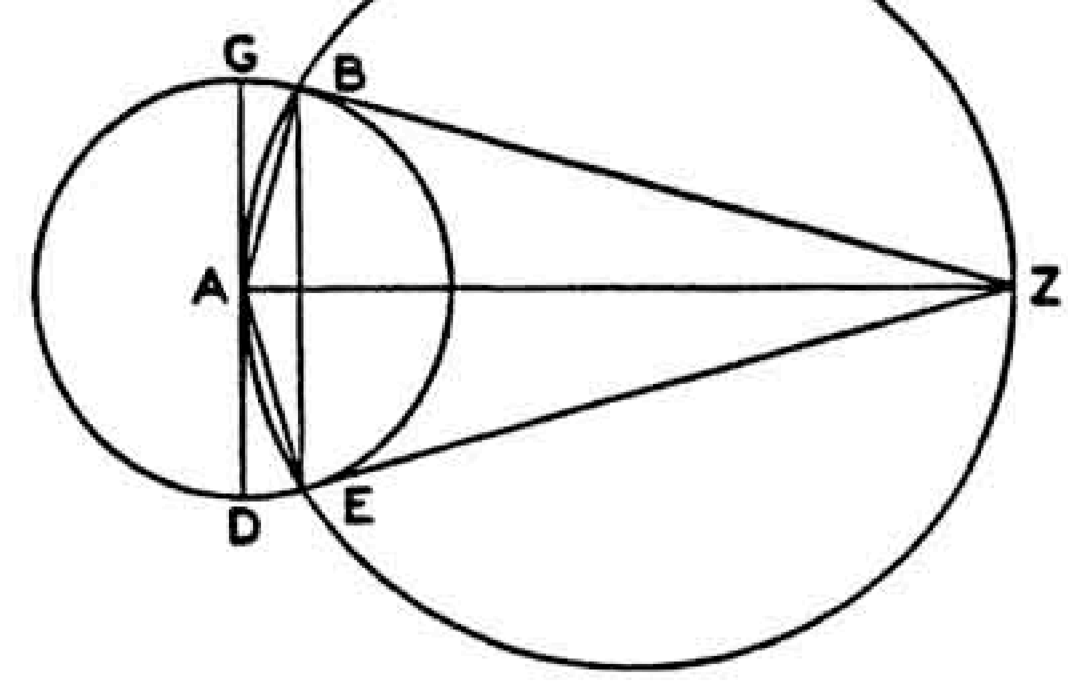


FIG. 28a.

A to Z let AZ be drawn. And from the point A let GD be drawn at right angles to AZ in both directions; so GD is the diameter of the circle. And around AZ let the circle $ABZE$ be inscribed, and let AB, BZ, ZE, and EA be drawn. So, ZB and ZE touch the circle, since the angles at the points B and E are right angles. Now, since from a certain point Z the rays ZB and ZE have fallen upon the circumference of the circle, the part of the circle, BE, will be seen. But $GBED$ is a semicircle. Hence, BE is less than a semicircle.

But this theorem was concerned with cones and cylinders. For, if from the points B and E the sides of the cylinders rise at right angles, the rays will touch them at the points where they fall, and the part of the vision indicated by $BGDE$ will be cut off, while the part of the semicircle indicated by BE will be seen. So, the same lesser part of cones also will be seen.

When the eye is moved nearer the cylinder, the part of the cylinder reached by the rays is less, but a greater part will appear to be visible. (Fig. 29.)

Let there by a cylinder, the base of which is the circle BG, and let A be the center, and let E be the eye, from which let EA be drawn to the center, and let the rays fall, EB and EG, and from the points B and G let the sides of the cylinder, GZ and BD, rise at right angles. Now, according to what has been said before, $DBGZ$ is less than half a cylinder; and it is seen by the eye, E. Now let the eye, T, be moved nearer. I say that the part reached by the eye, T, seems to be greater than $ZBGD$, but is actually less. Let the rays fall, TK and TL, and from the points K and L

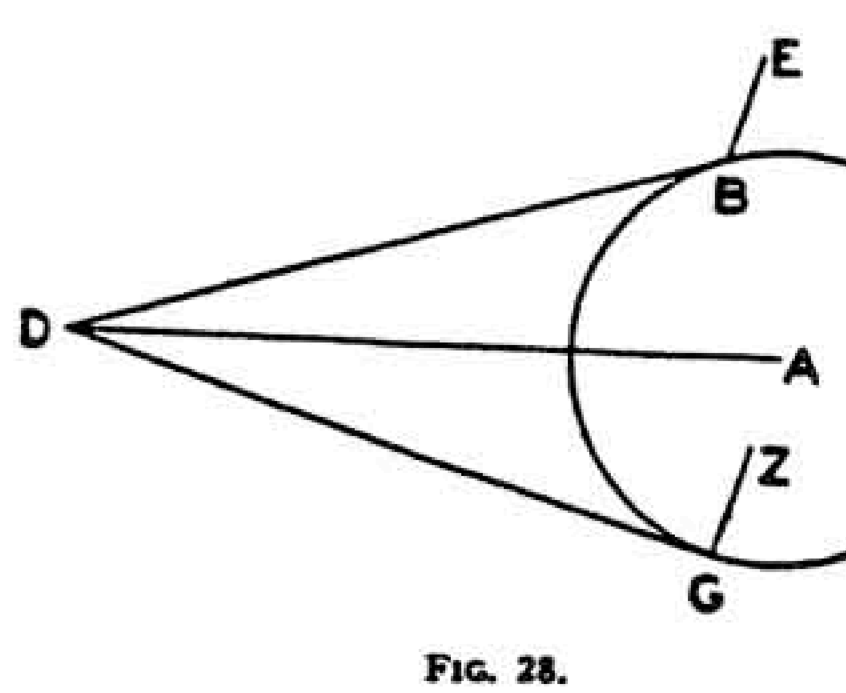


FIG. 28.

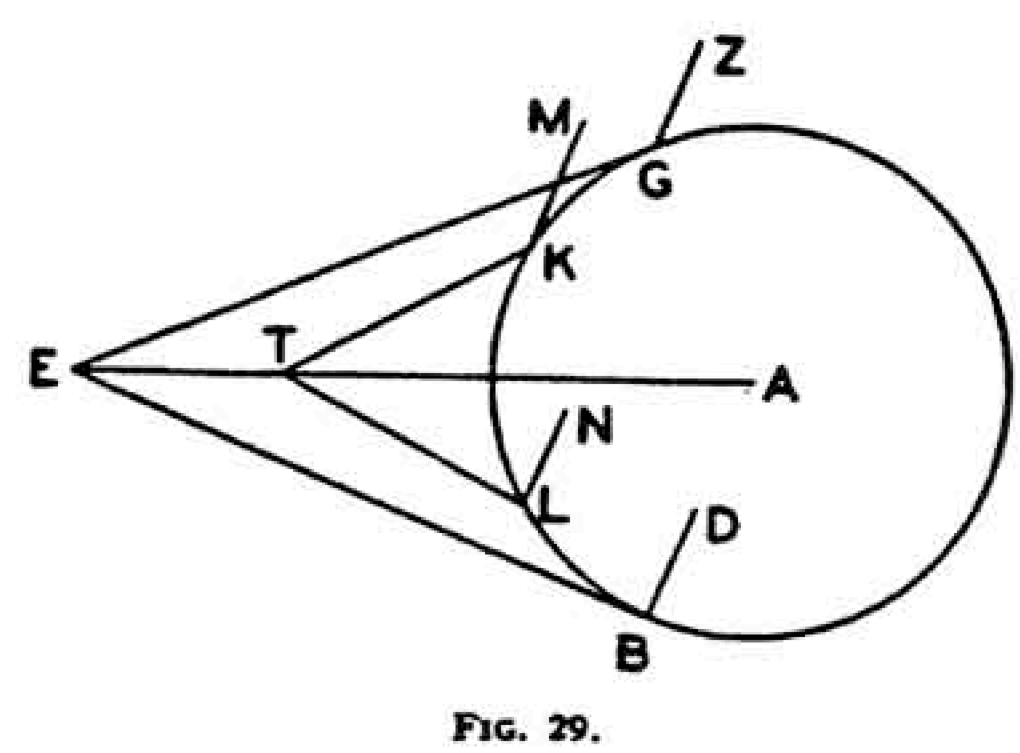


FIG. 29.

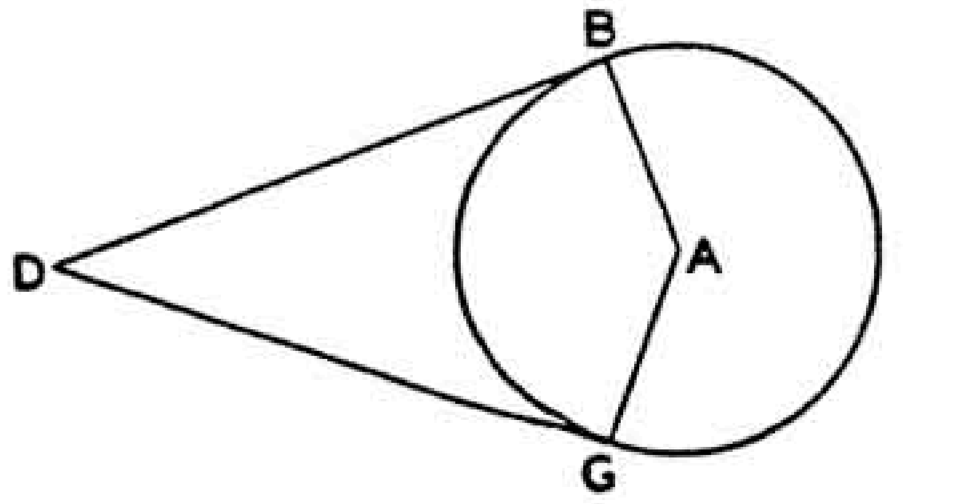


FIG. 30.

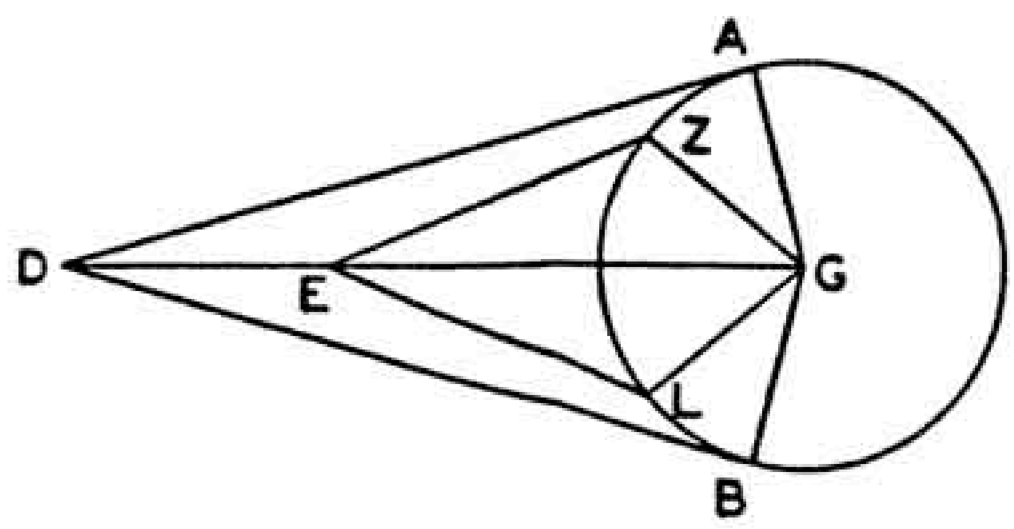


FIG. 31.

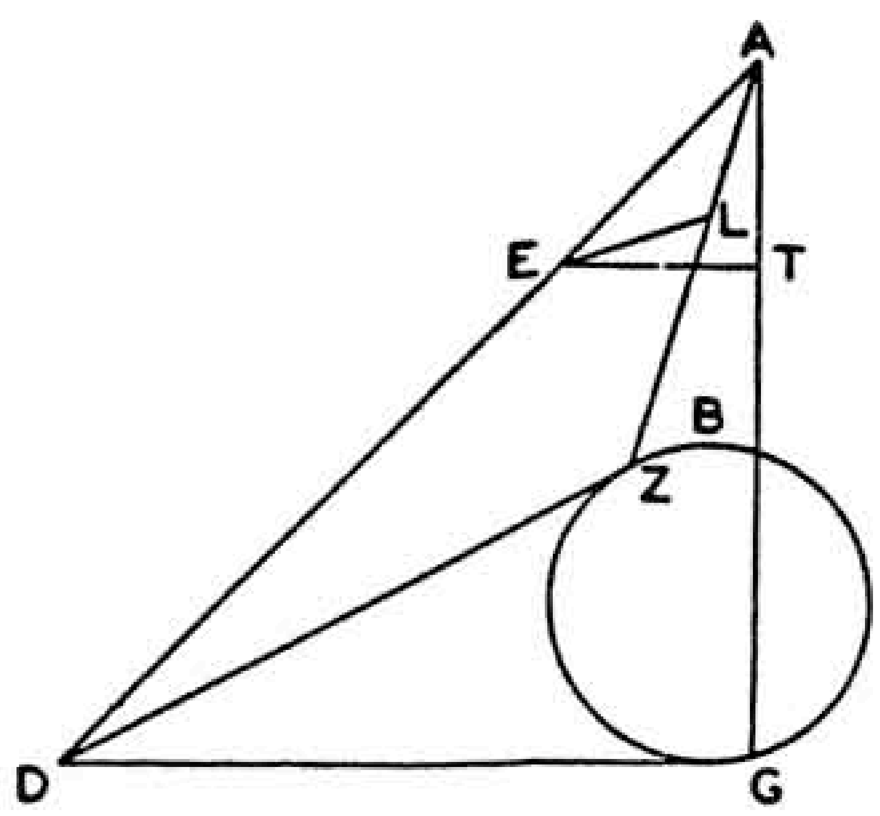


FIG. 32.

let the sides of the cylinder rise at right angles, KM and LN. Now the part of the cylinder, $MKLN$, will be seen by the rays TK and TL. But $ZGBD$ was seen by the rays EB and EG. And $ZGBD$ is greater than $MKLN$; but it seems to be less, since the angle at T is greater than the angle at E.

If a cone having a circle as its base and the axis at right angles to the base is seen by one eye, less than half the cone will be seen. (Fig. 30.)

Let there be a cone, the base of which is the circle BG, and the apex the point A, and let D be the eye, from which let the rays fall, DB and DG. And since the rays DG and DB have fallen, touching BG, BG is less than a semicircle, according to what has been proved before. From the apex of the cone, A, let the sides of the cone, AB and AG, be drawn to the points B and G. So the part included by the straight lines, AB and AG, and the arc BG is less than half a cone, since BG is less than a semicircle. Less than half a cone will be seen.

If the eye is brought nearer on the same plane on which is the base of the cone, the part included within the vision will be less, but a larger part will seem to be visible. (Fig. 31.)

Let there be a cone, the base of which is the circle AB, and the apex the point G, and let the eye be D, and let G be taken also as the center of the circle, and let the straight line, DG, be drawn, and let the rays fall, DA and DB, and let the sides of the cone be drawn, AG and GB. So, under the eye, D, and the lines of vision, DA and DB, the part of the cone indicated by ABG is included, and it is less than half a cone. Now let the eye come nearer, and let it be E, and let the rays fall, EZ and EL, and let the sides be drawn, ZG and GL. So again, under the eye E and the lines of vision, EZ and EL, the part of the cone ZGL is included. And ZGL is less than ABG; but it seems to be more, since the angle ZEL is greater than the angle ADB.

And it is clear that in the case of a cone, when seen by both eyes, the same things will occur as in the case of the sphere and the cylinder, when seen in the same way.

If rays fall from the eye upon the base of the cone, and if from the rays that fall and touch the cone straight lines are drawn from the points of contact over the surface of the cone to the apex of it, and if planes are produced through the lines that have been drawn and that fall from the eye upon the base of the cone, and if the eye is placed upon their point of contact, that is, upon the common section of the planes, the part of the cone that is seen will be quite the same, if the eye remains upon a plane parallel to the original plane. (Fig. 32.)

Let there be a cone, the base of which is the circle BG and the apex the point A, and let the eye be D, from which let the rays fall, DZ and DG, and from the points of contact, Z and G, let the sides of the cone, ZA and GA, be drawn to the apex of the cone, and let the plane through DZ and ZA, and the one through GD and GA be produced. Thus it will make a straight line the common section. Let it be AED. I say that, if the eye is moved upon the line AED, the same part of the cone will be seen, as much as was seen by the rays DG and DZ. For upon the line AED let the eye rest at E, from which let rays fall upon the cone. They will reach AZ and AG, since the eye rests upon a parallel plane, and the rays travel along straight lines. For, if they fall outside of AG and AZ, the rays will be broken; which is impossible. So, let ET and EL stand. Now, since the rays move upon a parallel plane in straight lines, and since things seen at equal angles appear equal, and since all the parallel rays upon the straight line AED contain equal angles, an equal part of the cone will be seen.

Again, if the eye is shifted from a low position, when it is raised, the part of the cone that is seen will be greater, but will seem to be less, but, on the other hand, if the eye is lowered, the part seen will be less, but will seem to be greater. (Fig. 33.)

Let there be a cone, the base of which is the circle BG, and the apex the point A, and let BA and AG be the sides

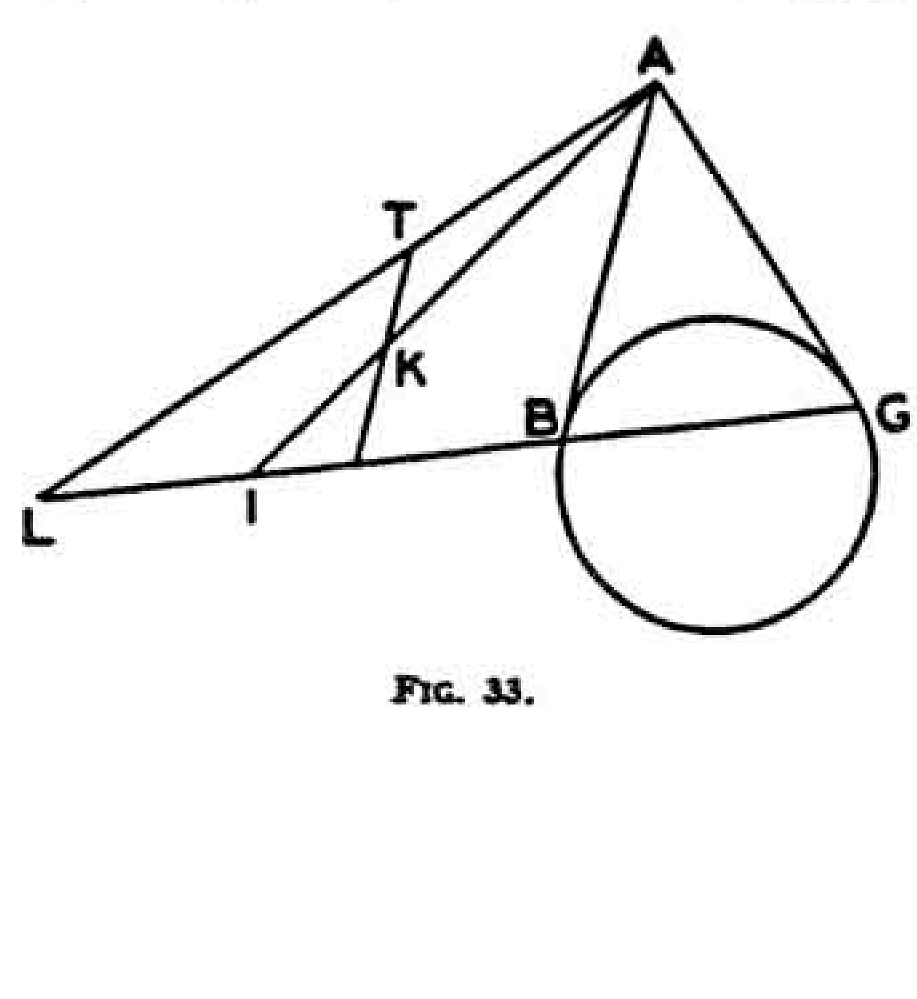


FIG. 33.

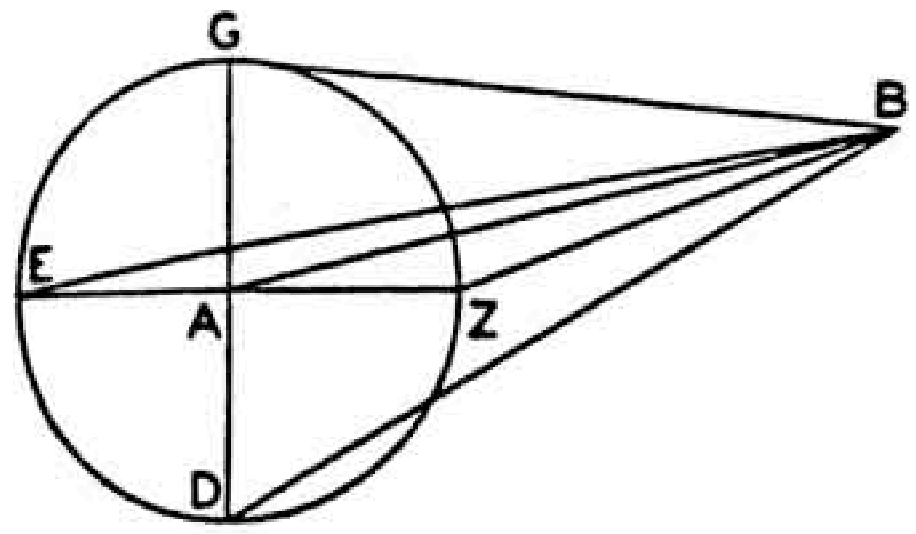


FIG. 34.

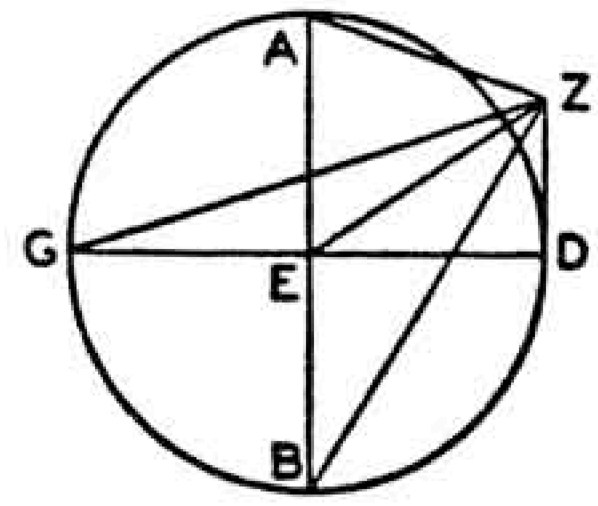


FIG. 34a.

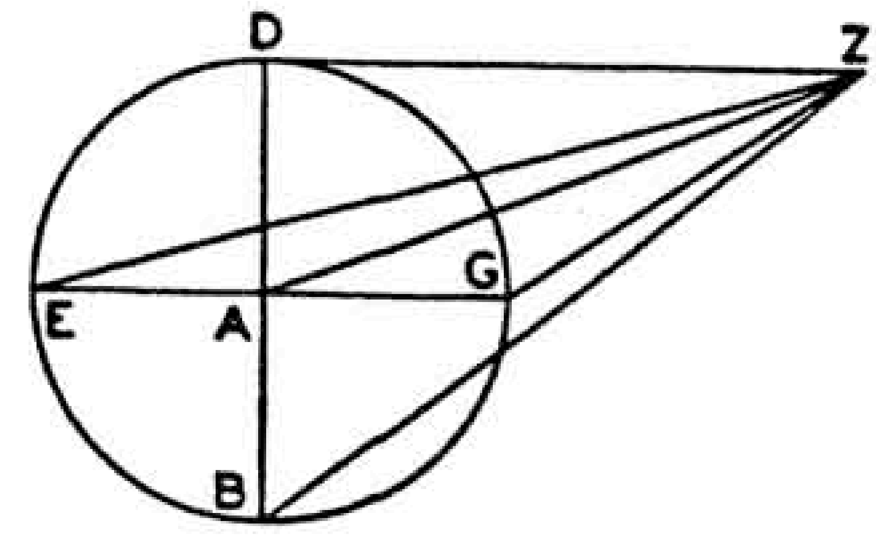


FIG. 34b.

of the cone. Let B and G be joined, and let BL continue the line of BG, and through the point T taken at random let TK be drawn parallel to AB. I say that the part of the cone that is seen will be greater, but will seem to be less, if the eye is placed at the point T than if it is placed at K. Let A and K be joined and A and T, and let AT be produced to L and AK to I. So, if the eye is placed at L and then at I, the parts of the cone that are seen will seem unequal, and the part seen from L will be greater, and the part seen from I, though less, will seem greater. But the part seen from L is equal to the part seen from T, and the part seen from I is equal to the part seen from K, as has been shown before. So, if the eye is placed at T, the part of the cone seen will be greater than if the eye is placed at K, but it will seem to be less.

If a straight line is erected from the center of a circle, at right angles to the plane of the circle, and the eye is placed upon this, all the diameters crossing upon the plane of the circle will appear equal. (Fig. 34.)

Let there be a circle, the center of which is the point A, and from it let the line AB be erected at right angles to the plane of the circle, and upon this let the eye B rest. I say that the diameters will appear equal. Let there be two diameters, GD and EZ, and let the lines be drawn, BG, BE, BD, and BZ. Now, since ZA is equal to AG, and AB is common to them, and the angles are right angles, then the base ZB is equal to the base BG, and the angles at the bases are equal. So the angle at ZB, BA is equal to the angle at AB, BG. Similarly, also the angle at EBA is equal to the angle at ABD. Then the angle at GB, BD is equal to the angle at EB, BZ. And the parts seen at equal angles appear equal. Then GD is equal to EZ.

And if the line drawn from the center is not at right angles to the plane, but is equal to the line from the center (that is, the radius of the circle), all the diameters will appear equal. (Fig. 34a.)

Let there be a circle, $ABGD$, and in it let two diameters be drawn, AB and GD, and, leading from the point E, let there be a line, ZE, upon which is the eye Z, not at right angles, but equal to each one of the lines from the center, and let the rays be drawn, ZA, ZG, ZB, and ZD. Now, since BE is equal to EZ, but also EA is equal to EZ, the three lines, EZ, EA, and EB, are equal. So the semicircle drawn in the plane through AB and EZ about the diameter AB will go through Z. So the angle at AZ, ZB is a right angle. Similarly, also the angle at GZ, ZD is a right angle. The right angles are equal, and things seen at the same angles appear equal. So AB will appear equal to GD.

But let AZ be not equal to the line from the center (that is, the radius), and not at right angles to the plane of the circle, but let it make equal the angles DAZ and ZAG, and the angles EAZ and ZAB. I say that even so the diameters making the equal angles will appear equal. (Fig. 34b.)

For, since GA, AZ are equal to ZA, AD, and since BA, AZ are equal to ZA, AE, and since the angles are equal, then the base DZ is equal to the base ZG; so that also the angle DZA is equal to the angle AZG. Now similarly we shall show that also the angle EZA is equal to the angle AZG. So, the whole angle DZB is equal to EZG. So that also the diameters DB and EG will appear equal.

But if the line falling from the eye to the center of the circle is not at right angles to the plane of the circle nor equal to the radius and does not enclose equal angles, the diameters with which it makes unequal angles will appear unequal. (Fig. 35.)

Let there be a circle $ABGD$, and let two diameters be drawn, AG and BD, cutting each other at right angles at the point E, and let the line ZE, starting from the point E, upon which rests the eye, be neither at right angles to the plane nor equal to the radius nor, with AG and DB, enclosing right angles. I say that the diameters AG and DB will appear unequal. For let ZG, ZA, ZD, and ZB be drawn. So the line EZ is either greater than the radius or less. On this account either the angle at DZ, ZB is greater than the angle at GZ, ZA, or the angle at GZ, ZA is greater than the angle at DZ, ZB, as we shall next prove. Thus the diameters will appear unequal.

Theorem. Let there be a circle, and let the center of it be the point A, and let B be the eye, and from this let a perpendicular drawn to the circle not fall upon the center A, but outside, and let this be BG, and from A let AG be drawn to G, and from A let AB be drawn to B. I say that of all the angles formed by the straight lines drawn through A and making an angle with the line AB, the least is the angle at GA, AB. (Fig. 35a.)

FIG. 35.

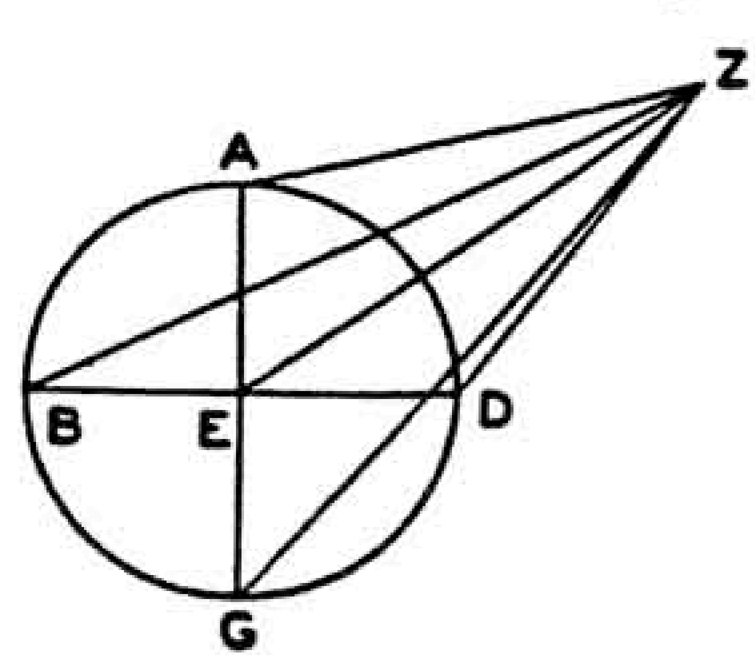
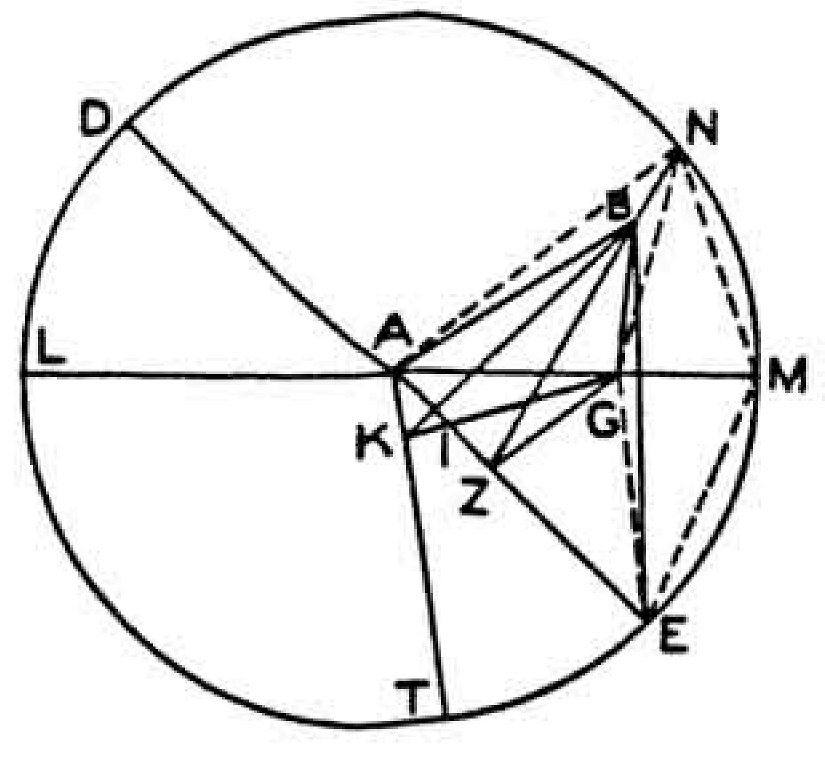


FIG. 35a.



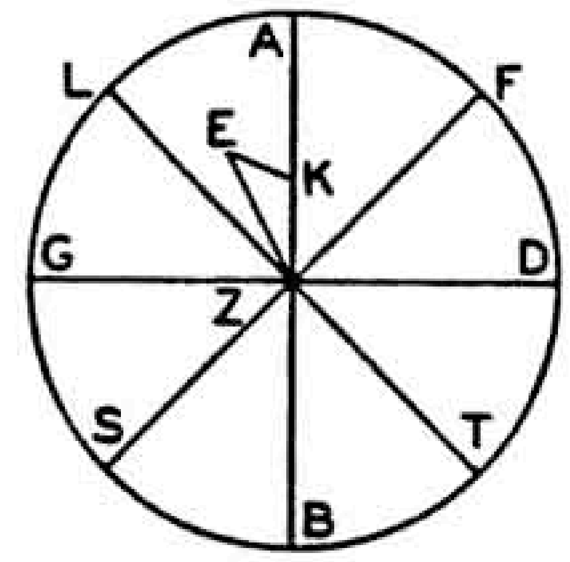


FIG. 35b.

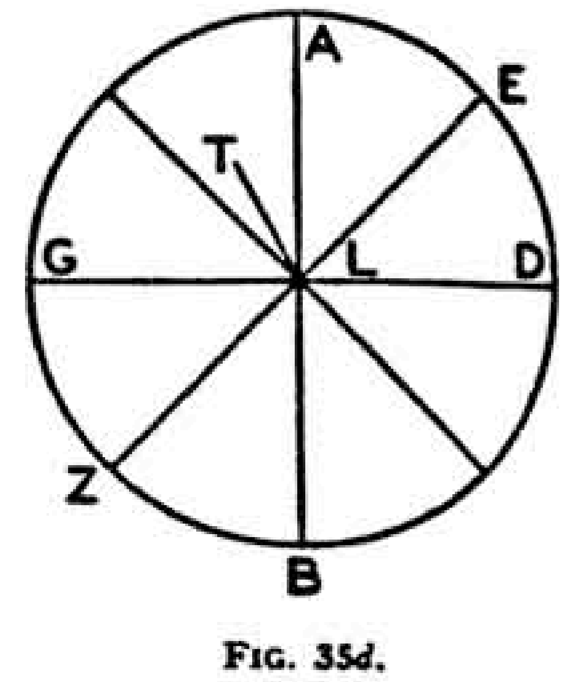


FIG. 35d.

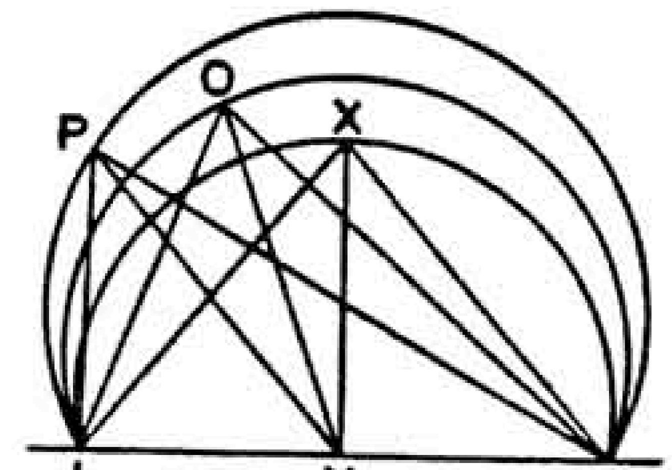


FIG. 35c.

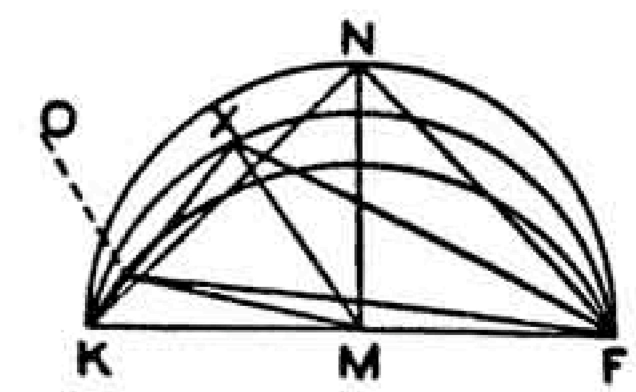


FIG. 35e.

Let the straight line DAE be drawn through A. I say that the angle GAB is less than the angle EAB. For from G to DE let the perpendicular GZ be drawn in the plane, and let B and Z be joined. Then BZ also is a perpendicular upon DE. Now, since the angle GZA is a right angle, the angle AGZ is less than a right angle. And the greater side subtends the greater angle. So, AG is greater than AZ. But the angle AG, GB, and the angle BZ, ZA, are right angles; so that GB and BZ are unequal. And, therefore, the angle ZA, AB is greater than the angle GA, AB. Now, similarly, it will be proved that of all the angles formed by the straight lines drawn through A and making an angle with the straight line AB, the least is the angle GA, AB.

And it is clear that, if also some other straight line is drawn through A, as AT, which is farther than AZ from AG, the angle BAT will be greater than the angle BAZ. For again, if the perpendicular GK is drawn to AT, B and K being joined will be similarly perpendicular to AT. And, since AI is greater than AK (for it subtends the right angle AKI), AZ is much greater than AK. And the angles BZA and BKA are right angles. So BZ is less than BK, on account of the fact that the distances from BZ, ZA and the distances from BK, KA, are equal to the distance from BA, and equal to one another, and again the angle BAK is greater than the angle BAZ. And of all the angles made with BA by the lines drawn through A, the greatest is the angle BAL, GA being produced to L, since also the angle BAG is less than all. And the lines are equal that are equally distant on either side from MA, the line which forms the least angle with BA. For, let MN be equal to EM, and let EM, MN, EG, GN, BA, BN, and AN be drawn. Now, since MN is equal to ME, and MG is common, and they form equal angles, then EG also is equal to GN. And GB is a common perpendicular. So, EB is equal to BN. But also EA is equal to AN; and AB is common. And the angle EAB is equal to the angle NAB.

Let there be a circle, $ABGD$, the center of which is Z, and in this let straight lines be drawn from A, B, G, and D, cutting each other at right angles, and let E be the eye, from which let there be a line to the center, joined at right angles to GD, but with AB let it form any angle; and let this line, EZ, be longer than the radius of the circle. I say that the diameters AB and GD will appear unequal, that GD will appear longer and AB shorter, and always the one nearer to the shorter one will appear shorter than the one farther away, and only two diameters will appear equal, those equally distant on either side of the shorter one. (Figs. 35b and 35c.)

For, since GD is at right angles to both AB and EZ, also all the planes produced through GD are at right angles to the planes through EZ and AB; so that it is true also of the plane of the circle lying below, upon which is GD. So, let a perpendicular be drawn from the point E to the plane below. It falls upon the common section of the planes, AB. Now let EK also fall, and let IM, equal to the diameter of the circle, be drawn and let it be cut in two at the point N, and from N, at right angles to IM, let the straight line NX be erected, and let NX be equal to EZ. Now the arc inscribed around IM and passing through X is greater than a semicircle, since the line NX is longer than either IN or NM. Let the arc be IXM, and let XI and XM be joined. The angle at X formed by the straight lines IX and XM is equal to the angle at the point E formed by E and by GD. On the straight line IN and at the point N let the angle IN, NO be formed, equal to the angle LZ, ZE, and let NO be equal to EZ, and let IO and OM be joined, and let the arc IOM be inscribed around the triangle IOM. Now the angle at the point O will be equal to the angle at E, formed by LET. Again on the straight line IN and at the point N upon it let the angle IN, NP be formed equal to the angle AZE, and let NP be equal to EZ, and let IP and PM be joined, and around the triangle IPM let the arc of a circle IPM be inscribed. Now the angle at the point P will be equal to the angle AEB. Now, since the angle at X is greater than the angle at O, but the angle at the point X is equal to the angle GED, and the angle at O is equal to the angle LET, GD will appear longer than LT. Again, since the angle at the point O is equal to the angle LET, and the angle at P is equal to the angle AEB, and since the angle at O is greater than the angle at P, also the angle LET is greater than the angle AEB. So LT will

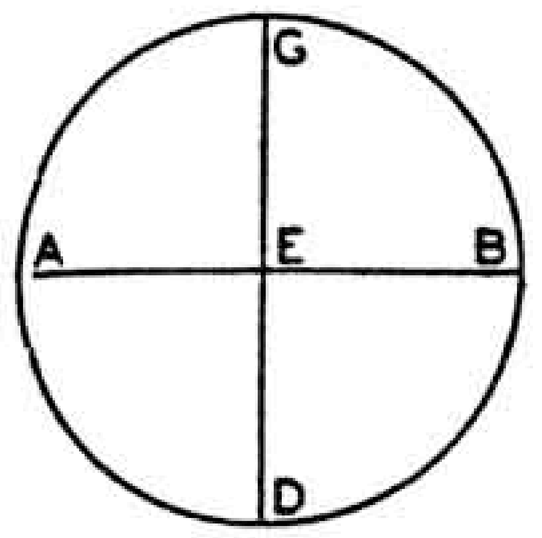


Fig. 36.

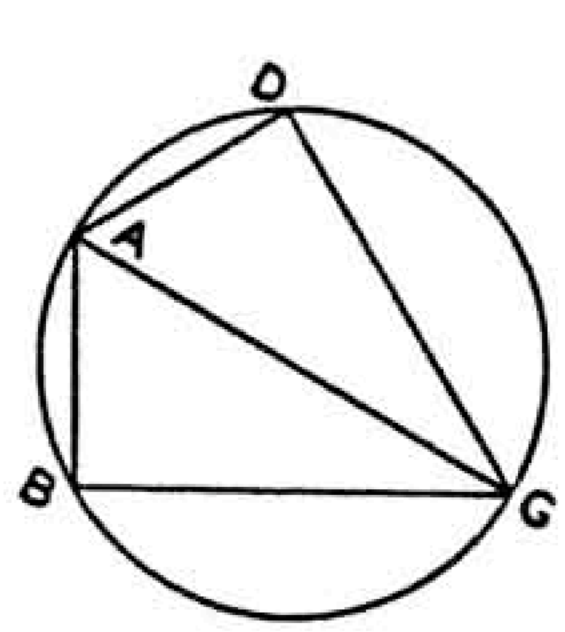


Fig. 37.

appear longer than AB. Now of all the straight lines drawn through Z and making angles with EZ, GD will seem to be the longest and AB the shortest, because, of the angles formed at E, GED is the greatest and AEB is the least. And only one other, FES, will be proved equal to the angle LET, AF, which is equal to LA, being excluded, and FZ being joined and produced to S. And this is clear from the angles at X, O, and P. For the least of these is P, since the angle PNI is equal to the least angle, EZA, and the greatest is X, on account of the fact that NX is a perpendicular, the longest of the straight lines drawn through N in the arc IXM, and because the arc IXM falls above the equal line, and because X is the innermost point and P the outermost, since no angle is less than PNI. And the angle EZF, being equal to the angle EZL, as has been proved, also the adjacent angle, EZS, is equal to the angle EZT, that is, to the angle ONM. So that each of the angles, FES and LET, is equal to the angle at O. So, LT will appear equal to FS.

Let the line drawn from the eye to the center be less than the radius. Now about the diameters the reverse is true; for the one appearing longer before now will appear shorter, and the shorter one will appear longer. (Figs. 35*d* and 35*e*.)

Let there be a circle, $ABGD$, and let two diameters be drawn, AB and GD, cutting each other at right angles, and let another diameter, EZ, be drawn at random, and let the eye be T, from which let the line LT be drawn to the center, and let it be shorter than either of the radii. And let KF be drawn, equal to the diameter of the circle, and let it be cut in two at M, and from the point M let MN be erected at right angles, and let MN be equal to TL, and let an arc of a circle, NKF, be inscribed about KF and the point N. Now it is less than a semicircle, since MN is less than the radius. Now the angle at N formed by KN and FN will be equal to the angle at T formed by GT and TD. Again, let the angle KMX be equal to the angle ELT, and let MX be equal to LT, and let an arc, KXF, be inscribed about KF and the point X. Now the angle at the point X formed by KXF is equal to the angle at T formed by ZTE. Again, let the angle KM, MO be equal to the angle AL, LT, and let MO be equal to LT, and about KF and O let an arc be inscribed. Now the angle at O formed by KOF will be equal to the angle at T formed by ATB. So, since the angle at O is greater than the angle at X, and the angle at O is equal to the angle at T formed by ATB, and the angle at X is equal to the angle at T formed by ETZ, AB will appear longer than EZ. Again, since the angle at T formed by ET, TZ is greater than the angle at T formed by GTD, EZ will appear longer than GD.

The wheels of the chariots appear sometimes circular, sometimes distorted. (Fig. 36.)

Let there be the wheel $ABGD$, and let the diameters be drawn, BA and GD, cutting each other at right angles at the point E, and let the eye not be on the plane of the circle. Now, if the line drawn from the eye to the center is at right angles to the plane or is equal to the radius, all the diameters will appear equal; so that the wheel appears circular. But if the line drawn from the eye to the center is not at right angles to the plane or equal to the radius, the diameters will appear unequal, one longer and one shorter, and to every other one which is drawn between the longer and shorter diameters only one other will appear equal, the one drawn to the opposite side; so that the wheel appears distorted.

There is a place where, if the eye remains in the same position, while the thing seen is moved, the thing seen always appears of the same size. (Fig. 37.)

Let the eye be A, and the size of the thing seen be indicated by BG, and from the eye let rays fall, AB and AG, and about ABG let a circle, ABG, be inscribed. I say that there is a place where, if the eye remains in the same position, while the thing seen is moved, the thing seen always appears of the same size.

For let the object be moved and let it be indicated by the line DG, and let AD be equal to AB. Now, since BA is equal to AD and BG to GD, also BAG is equal to DAG. For they are upon equal arcs; so that they are equal. So the thing seen will appear of the same size.

And the same thing will happen also if the eye should remain at the center of the circle, and the thing seen should move upon the arc.

There is a place where, if the position of the eye is changed, while the thing seen remains in the same place, the thing seen always appears of the same size. (Fig. 38.)

For let BG be the thing seen and let Z be the eye, from which let rays fall, ZB and ZG, and about the triangle BZG let the arc of a circle, BZG, be inscribed, and let the eye, Z, be shifted to D, and let the rays fall, DB and DG. Now the angle D is equal to the angle Z; for they are on the same arc. And things seen at the same angle appear equal. So BG will always appear of the same size, when the eye is shifted upon the arc BDG.

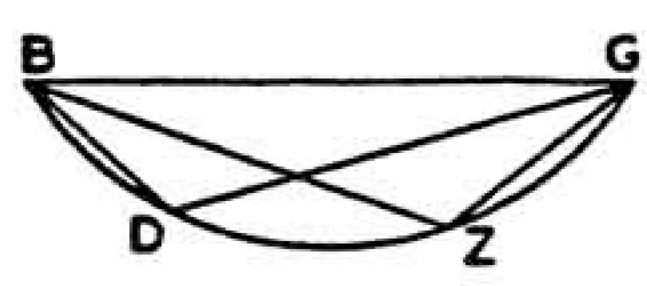


Fig. 38.

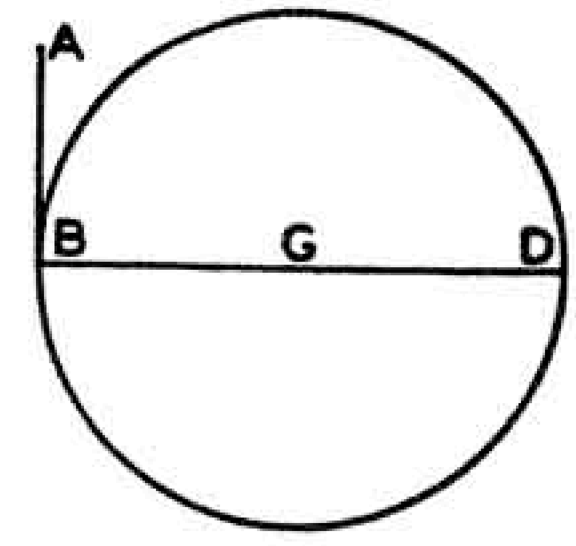


Fig. 39.

If some object is perpendicular to the plane lying below, and if the eye is placed upon some point of the plane, and if the thing seen is shifted upon the circumference of a circle which has the eye as a center, the thing seen will always appear of the same size when it is moved to a position parallel to its original position. (Fig. 39.)

Let AB be some object that is seen, located at right angles to the plane, and let G be the eye. And let G and B be joined, and with G as the center and GB as the radius let the circle BD be inscribed. I say that if, upon the circumference of the circle, the object AB is moved, as seen by the eye G, AB will appear to be of the same size. For AB is a perpendicular and makes a right angle with BG, and all the straight lines falling from the center G upon the circumference of the circle make equal angles. So the object seen will appear to be of the same size.

And if from the center G a straight line rises at a right angle and the eye is placed upon this, and the object seen is moved along the circumference of the circle, parallel to the straight line upon which is the eye, the thing seen will always appear to be of the same size.

But if the thing seen is not at right angles to the plane lying below, and, being equal to the radius, is moved upon the circumference of the circle, it will appear sometimes equal to itself, sometimes unequal, if it is shifted to a position parallel to its original position. (Fig. 40.)

Let AD be a circle and let the mark D be taken on its circumference, and let the line DZ, which is equal to the radius, rise not at right angles to the circle, and let E be the eye. I say that DZ, if it is moved on the circumference of the circle, will appear sometimes equal, sometimes greater, sometimes less. Now through E, which is the center, let GE be drawn, parallel to DZ, and let EG be equal to DZ. And from the point G let the perpendicular GL be drawn to the plane lying below and let it reach the plane at the point L. And let EL be connected and produced, and let it reach the circumference at the point A, and through A let AB be drawn parallel to GE, and let AB be equal to DZ. I say that AB, of all the straight lines moved upon the circumference of the circle, will appear the shortest. For let the straight lines, ED, GZ, GB, EB, and ZE, be drawn. Now, since GE is parallel and equal to AB, also EA is equal and parallel to GB. So, $AEGB$ is a parallelogram. For the same reasons also $EDZG$ is a parallelogram. And it remains to prove that the same thing appears smaller and greater. Now it is clear that the angle GEA is less than the angle GED, since it has been proved that of all the straight lines drawn through the center and making an angle, the angle GEA is the smallest. So it is smaller than the angle GED also. And the angle BEA is half the angle GEA; for BE is an equilateral parallelogram; and the angle ZED is half the angle GED; for ZE also is an equilateral parallelogram. And so the angle BEA is less than the angle ZED. So that also the object AB will appear less than the object DZ.

And it is clear from the thesis before proved that it will appear least at the point A, and greatest when diametrically opposite to the point A, and equal at an equal distance on either side of the point A.

But if the thing seen is at right angles to the plane lying below, and the eye moves upon the circumference of a circle which has as its center the point at which the object meets the plane, the thing seen will always appear of the same size. (Fig. 41.)

Let the object seen be AB, at right angles to the plane lying below, and let the eye be G. And with B as a center and the radius BG, let the circle GD be inscribed. I say that if G moves upon the circumference of the circle, AB will always appear the same. And this is clear. For all the rays from the point G falling upon AB fall at equal angles, since the angle at B is a right angle. So the thing seen will appear of the same size.

If the thing seen remains in its original position and the eye moves in a straight line that is oblique to the object seen, the object appears sometimes of the same size, sometimes of different size. (Fig. 42.)

Let AB be the thing seen and let E be the eye, and let GD be a straight oblique line, and let GA be a continuation of BA in a straight line, and let it meet DG at G, and let the eye be moved upon it. I say that AB appears sometimes of the same size, sometimes of different size. For, let GE be taken as a mean proportional of the lines BG and GA, and let E be the eye and let it be moved, and let it be on the same straight line at D. I say that the thing seen from E and D appears to be of different size. Let the straight lines AE, EB, AD, and BD be drawn, and about the triangle AEB let the arc AEB be inscribed, and let the angle formed

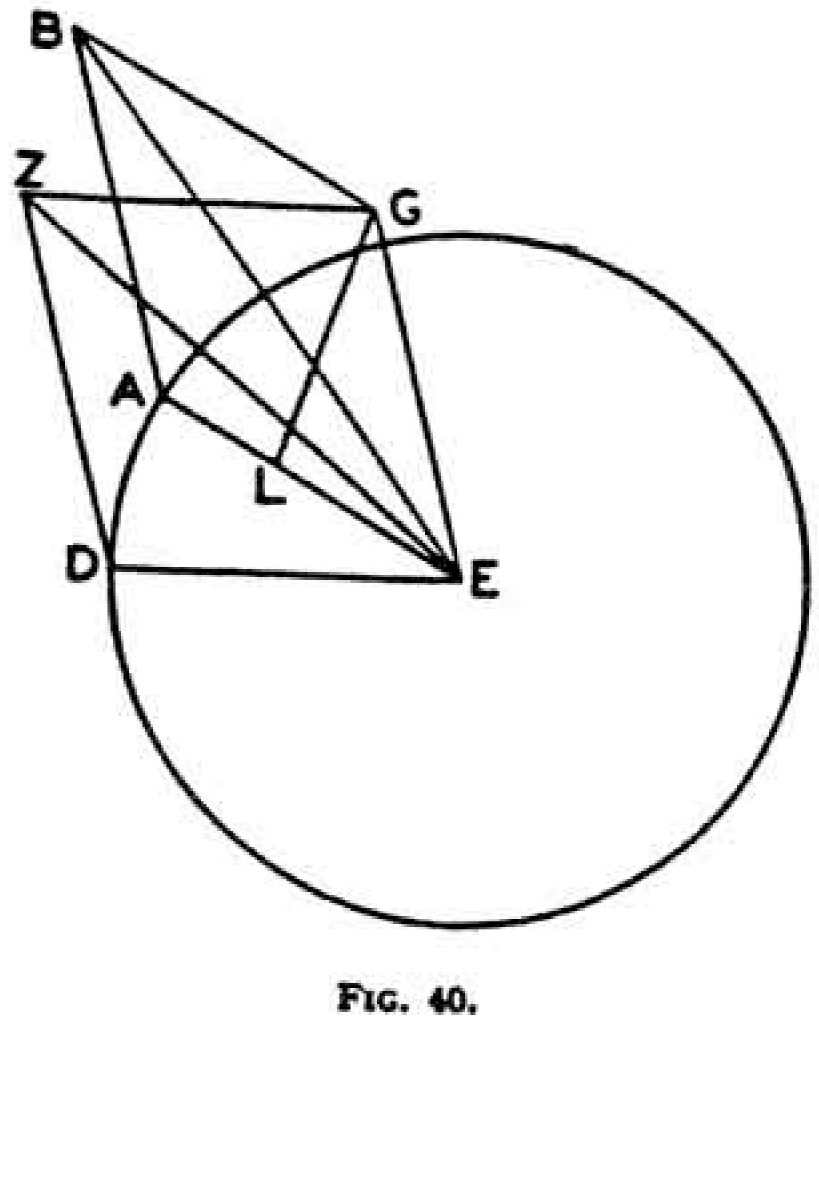


Fig. 40.

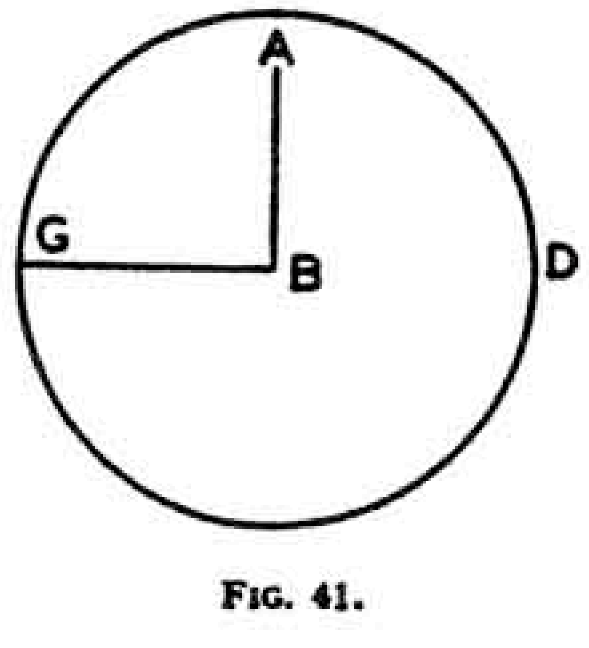


Fig. 41.

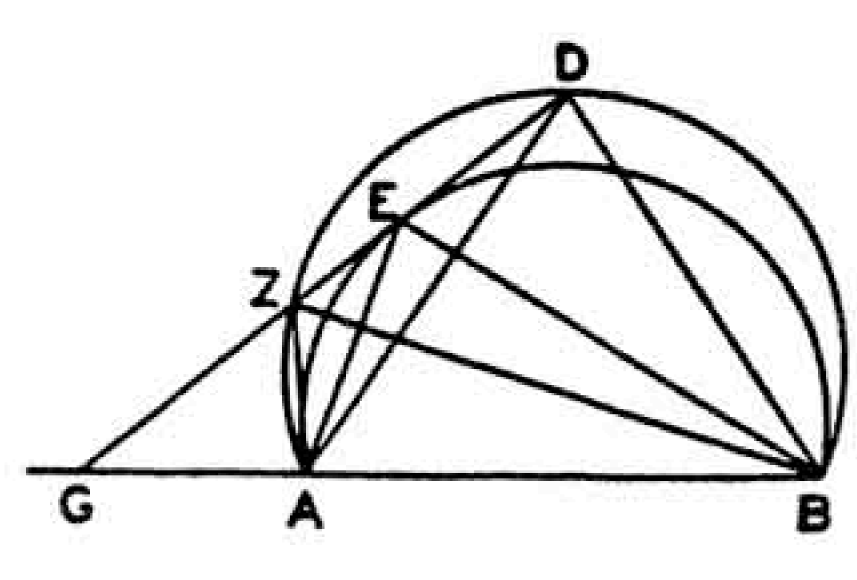


Fig. 42.

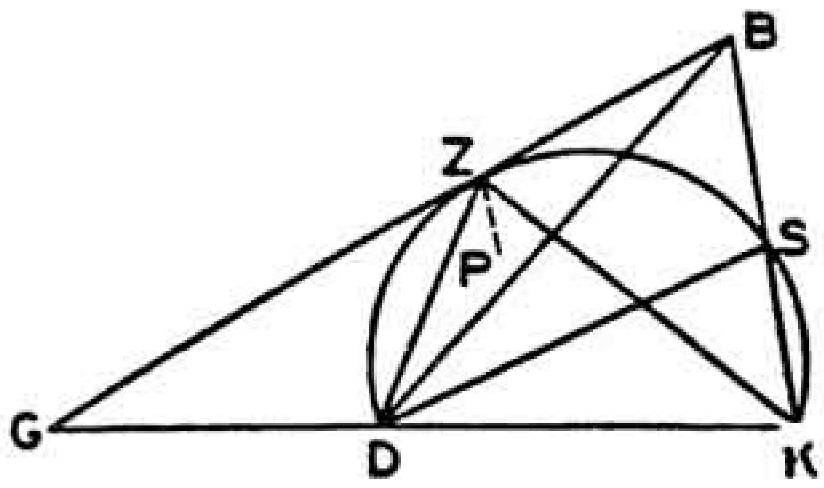


FIG. 42a.

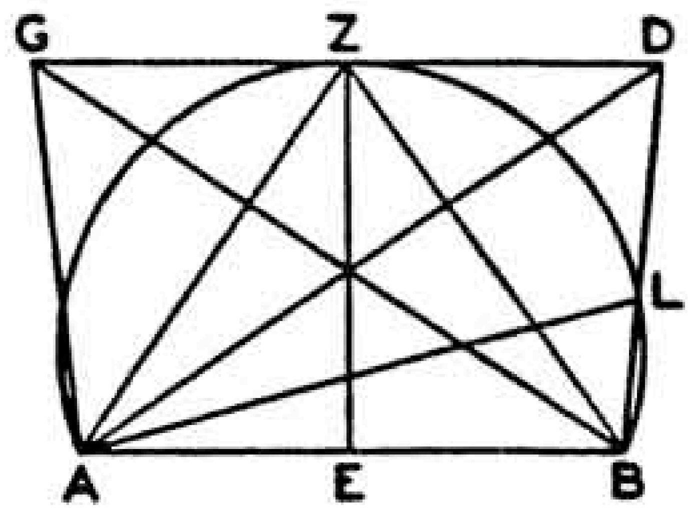


FIG. 43.

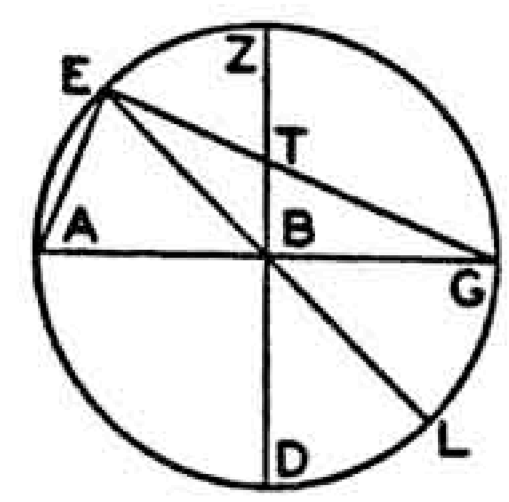


FIG. 44.

by GA and AZ be equal to the angle formed by GD and DB, and let BZ be joined. Now B, A, Z, and D are points in a circle. So, since the angle AEB is greater than the angle AZB, and the angle AZB is equal to the angle formed by AD and DB, since it is on the same arc, also the angle AEB is greater than the angle ADB. But under the angle ADB, AB is seen when the eye is at D, and under the angle AEB the same AB is seen when the eye is at E. So the thing seen appears of different size when the eye moves along the straight line ED. And it is clear also that when the eye moves along the line EG, the thing seen appears of different size, and that it appears largest from the position at E, but always larger in a position nearer to E on either of the straight lines, ED and EG, and of the same size at Z and D and places taken in like manner to them, on account of the fact that the angles are on the same arc.

In addition. (Fig. 42a.) For let KD be the thing seen and let BG be a straight line falling upon KD produced. Let GZ be taken as a mean proportional of GD and GK, and let ZK and ZD be drawn, and about KD let an arc be inscribed, which contains the angle KZD. Now it will touch the straight line BG, since as KG is to GZ, so is GZ to GD. Now let the eye be at the point B, and let DB and BK be added. And let S and D be joined. So the angle P is equal to the angle S; for they are on the same arc. And the angle S is greater than the angle B; and so the angle P is greater than the angle B. Thus the eye being at Z, KD appears greater than when the eye is at B.

And the same thing will happen, even if the straight line is parallel with the object seen. (Fig. 43.)

Let AB be the object seen and let it be cut in two at the point E, and from E let EZ be erected at right angles to AB, upon which line let the eye rest at Z, and let the straight lines ZA and ZB be drawn, and about the triangle AZB let the arc AZB be inscribed, and let ZD be drawn through Z parallel to AB, and let the eye move to D, and let the rays fall, DA and DB. I say that from the points D and Z things will appear of different size. Let A and L be joined. Now, since the angle AZB is equal to the angle ALB, but the angle ALB is greater than the angle ADB, the angle AZB also is greater than the angle ADB. And AB is seen under the angle AZB when the eye is at Z, and similarly under the angle ADB when the eye is at D. So the thing seen appears of different size from the points D and Z.

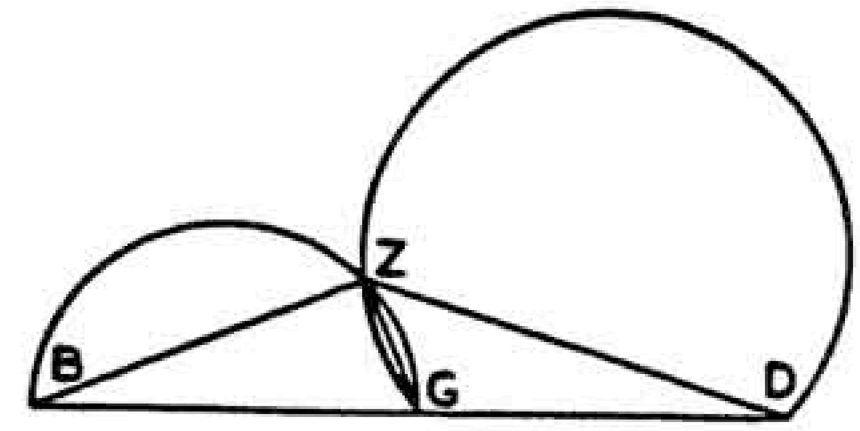
And if ZG is drawn equal to DZ, the object appears smaller from G than from Z, but from G and D it appears the same.

There are places where, if the eye is moved to them, objects that are of equal size and close to one another appear sometimes of the same size, sometimes of different sizes. (Fig. 44.)

Let T be the eye, and let AB and BG be the objects, and from B let BZ be drawn at right angles, and let it be produced to D. Now it is clear that if the eye is placed on any part whatever of the line ZD, AB and BG will appear equal. Now let the eye be moved and let it be E. I say that from E the objects appear of unequal size. Let the rays fall, EA, EB, and EG, and around the triangle AGE let the circle $AEDG$ be inscribed, and let BL be a continuation of EB. Now, since the arc AD is equal to the arc DG, and since the arc ADL is greater than the arc LG, AB will appear greater than BG. And if the eye moves upon the line EL, the objects will appear similarly unequal; and if it is placed upon the parts of the circle outside the perpendicular, the objects appear unequal, and if it is placed outside the circle, not in line with DZ, they appear unequal.

In addition. (Fig. 44a.) For let BG be equal to GD, and about BG let the semicircle BZG be inscribed, and about GD let GZD, more than a semicircle, be inscribed; and it is clear that it will cut the semicircle before mentioned. And it is possible to inscribe upon GD an arc greater than a semicircle. For if we assume a certain acute angle, it is possible for us to inscribe upon GD an arc of a circle containing an angle equal to the acute angle below, as (is clear) from the 33rd proposition of the third book of the Elements, and the arc imposed upon it will be greater than a semicircle, as (is clear) from the 31st proposition of the third book of the Elements. And let BZ, ZG, and ZD be drawn. So the angle in the semicircle is greater than the angle in the greater arc. And the things seen at a greater angle appear larger; so BG appears larger than GD. But it was equal. There is, then, a common place, where, if the eye is placed there, equal things appear unequal. But they will appear equal when . . . . (The remainder of this sentence is corrupt in the Greek text, and cannot be translated.)

FIG. 44a.



There is a certain place from which objects of unequal size appear equal. (Fig. 45.)

For let BG be greater than GD, and around BG let an arc greater than a semicircle be inscribed, and around GD an arc like the one around BG, that is, one containing an angle equal to the one in BZG. Now the arcs will cut each other. Let them cut each other at Z, and let ZB, ZG, and ZD be drawn. So, since the angles in the similar arcs are equal to each other, also the angles in the arcs BZG and GZD are equal to each other. And the things seen under equal angles appear equal. For if the eye is placed at the point Z, BG would appear equal to GD. But it really is greater. So there is a common place, from which objects of unequal size appear equal.

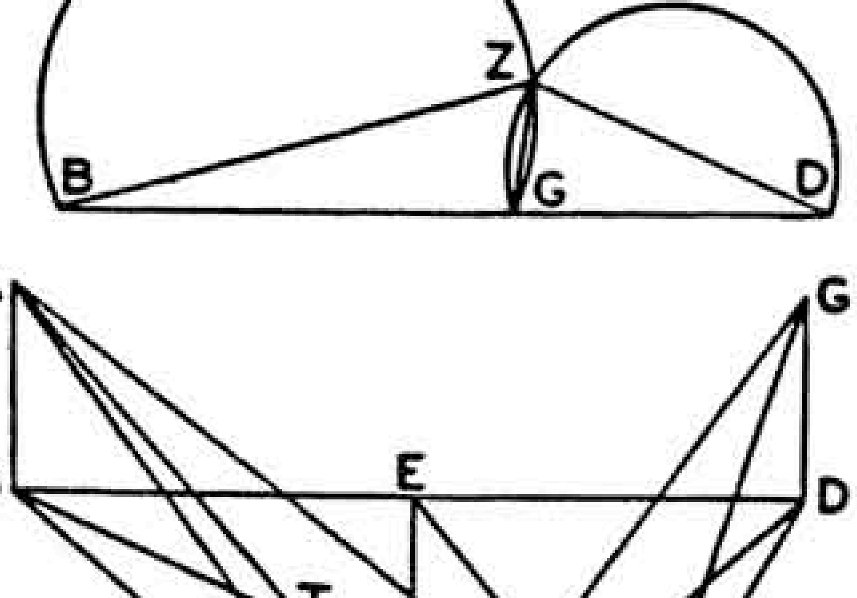
There are places to which the eye may be moved and objects of equal size that are perpendicular to the plane below appear sometimes of equal size, sometimes of unequal size. (Fig. 46.)

Let AB and GD be objects of equal size perpendicular to the plane below. I say that there is a place where, if the eye is located there, AB and GD appear equal. Let BD be drawn from B to D, and let it be cut in two at the point E, and from E let EZ be drawn at right angles to DB. I say that if the eye is placed upon EZ, AB and GD will appear equal. For let the eye rest upon EZ, and let Z mark the position of the eye, and let the rays fall, ZA, ZB, ZE, ZD, and ZG. Now the straight line ZB is equal to ZD. But also AB is equal to GD. So the two lines, AB and BZ, are equal to the two, GD and DZ. And they contain equal angles; for the angle BZA is equal to the angle DZG. So AB and GD will appear equal.

I say, however, that they may appear also unequal.

Now let the eye move, and let it be indicated by L, and let L and E be joined, and let the rays fall, LB, LA, LG, and LD. So, LB is longer than LD. From LB let BT be taken, equal to LD, and let A and T be joined. Now the angle BTA is equal to the angle GLD. But the angle BTA is greater than the angle BLA, the exterior angle greater than the interior. And so the angle GLD is greater than the angle BLA. Therefore, GD will appear greater than AB.

Fig. 45.



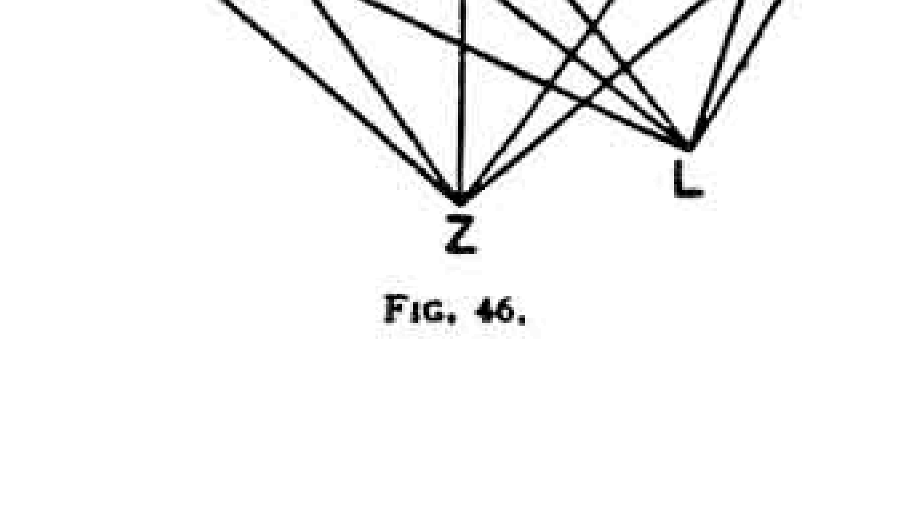
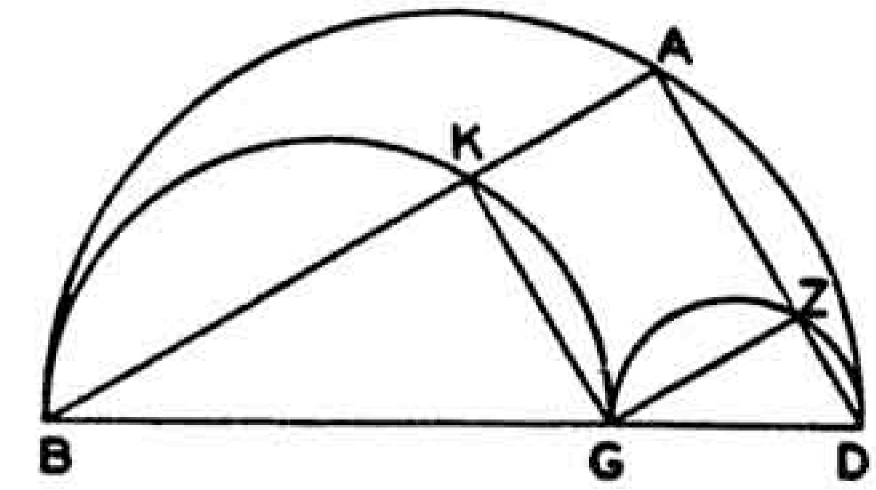


Fig. 46.

Fig. 47.



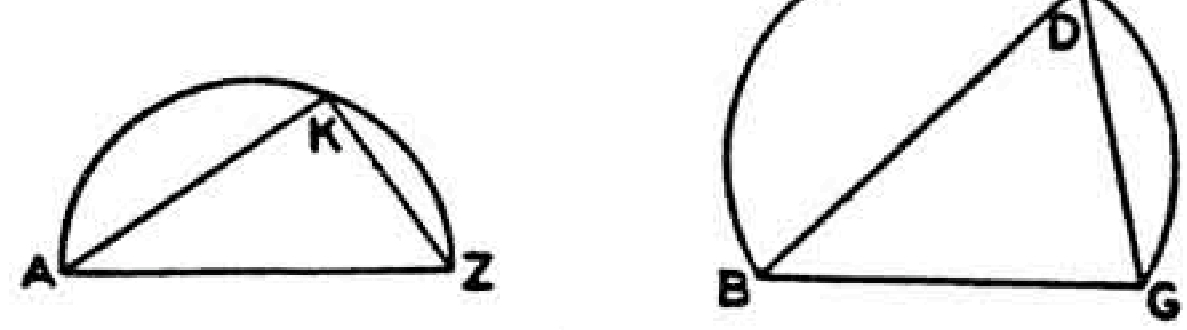


Fig. 48.

There are certain places where the eye may be located, and objects of unequal size placed in contact will together appear equal to each of the unequal objects. (Fig. 47.)

For let BG be larger than GD, and about BG and GD let semicircles be inscribed and another about the whole BD. So the angle in the semicircle BAD is equal to the angle in BKG; for each of them is a right angle. So BG appears equal to BD. And in like manner also BD appears equal to GD, when the eyes are upon the semicircles BAD and GZD. There are, then, certain places where two objects of unequal size, when placed in contact, together appear equal to each of the unequal objects.

To find places from which an object of equal size appears half or a quarter as large, or in general appears of a size proportionate to the cutting of the angle. (Fig. 48.)

Let AZ be equal to BG, and about AZ let a semicircle be inscribed, and in it let the right angle K be drawn; and let BG be equal to AZ, and about BG let an arc be inscribed which will contain half of the angle at K. So the angle K is double the angle D. So AZ appears twice the size of BG, when the eyes are upon the arcs AKZ and BDG.

Let AB be the object seen. I say that there are places where the eye may be located and the same thing appears sometimes half the size, sometimes the whole size, sometimes quarter size, and in general in the given proportion. (Fig. 49.)

About AB let the circle AEB be inscribed, so that AB is not the diameter, and let the center of the circle be taken and let it be G, upon which let the eye rest, and let the straight lines AG and GB be drawn. So AB is seen under the angle AGB. Now let the eye rest upon the circumference of the circle and let it be indicated by E, and let the rays fall, EA and EB. Now, since the angle AGB is double the angle AEB, from G, AB seems twice as large as when seen from E. Similarly also, it will appear a fourth as large if one angle is a fourth as great as the other angle, and (it will always be seen) in the given proportion.

If objects move at equal speed, and have their ends on the same side of a straight line which is at right angles to their course, as they advance toward a line drawn through the point where the eye is located, which is parallel to the straight line

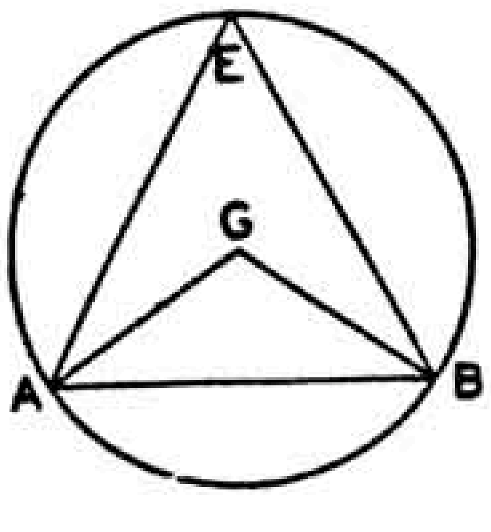


FIG. 49.

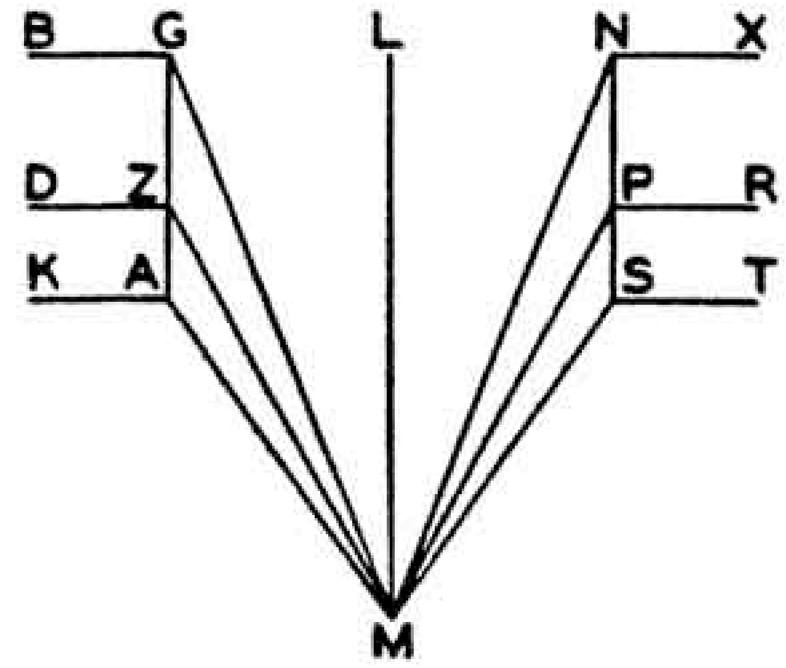


FIG. 50.

before mentioned, the one farther away from the eye will seem to be ahead of the nearer one, but when they have passed (the direct line of vision), the one that was in the lead will seem to follow, and the one that followed will seem to be in the lead. (Fig. 50.)

For let BG, DZ, and KA move with equal speed, having their ends, G, Z, and A, on the same side of a straight line, GA, which is at right angles to their course, and from the eye, M, let ML be drawn parallel to GA, and let MG, MZ, and MA be drawn. So BG seems to be in the lead and KA seems to follow, on account of the fact that, of the rays falling from the eye, MG, leading to G, seems to have diverged (to the right) more than the other rays. So BG will seem to be in the lead of the moving objects, as has been said. But when BG, DZ, and KA have passed (the direct line of vision), and have become NX, PR, and ST, let the rays fall, MN, MP, and MS. So NX seems to have moved in the direction of N (that is, toward the left), on account of the fact that the ray MN has been diverted to N more than the other rays; so ST has moved in the direction of T (that is, toward the right), on account of the fact that MS has been diverted toward T more than the other rays. So when BG, which seemed to be in the lead, has become NX, it will seem to follow, but when KA, which seemed to follow, has become ST, it will seem to be in the lead.

If, when several objects move at unequal speed, the eye also moves in the same direction, some objects, moving with the same speed as the eye, will seem to stand still, others, moving more slowly, will seem to move in the opposite direction, and others, moving more quickly, will seem to move ahead. (Fig. 51.)

For, let B, G, and D move at unequal speed, and let B move most slowly, and G at the same speed as the eye, K, and D more quickly than G. And from the eye, K, let the rays fall, KB, KG, and KD. So, G, moving with the eye, will seem to stand still, and B, left behind, will seem to move in the opposite direction, and D, which moves more quickly than these, will seem to move forward; for it will be more distant from these.

If, when certain objects are moved, one is obviously not moved, the object that is not moved will seem to move backward. (Fig. 52.)

For, let B and D move, and let G remain unmoved, and from the eye let the rays fall, ZB, ZG, and ZD. So B, as it moves, will be nearer to G, and D, receding, will be farther away; therefore, G will seem to move in the opposite direction.

When the eye moves nearer the object seen, the object will seem to grow larger. (Fig. 53.)

For, the eye being at Z, let BG be seen by the rays ZB and ZG, and let the eye move nearer to BG, and let it be at D, and let the same thing be seen by the rays DB and DG. So the angle D is greater than the angle Z; and things seen under a wider angle appear larger. Therefore BG will seem to be larger when the eye is at D, than when it is at Z.

When objects move at equal speed, those more remote seem to move more slowly. (Fig. 54.)

For let B and K move at equal speed, and from the eye, A, let rays be drawn, AG, AD, and AZ. So, B has longer rays than K extending from the eye. Therefore it will cover a greater distance, and, later, passing the line of vision, AZ, will seem to move more slowly.

An addition. (Fig. 54*a*.) For let two points, A and B, move on parallel straight lines, and let Z be the eye, from which let the rays fall, ZA, ZB, ZE, and ZD. I say that the more remote object, A, seems to move more slowly than B. For, since AZ and ZD form a smaller angle than ZB and ZE, BE appears greater than AD. (The following sentence is corrupt in the Greek text, and no satisfactory translation is possible.) Therefore, if we extend the ray ZE in a straight line, since in the case of objects moving at the same speed B reaches the ray ZE later than the things moving at the same speed, the more remote objects seem to move more slowly.

An addition. (Fig. 54*b*). Let two points, A and B, move evenly on the parallel straight lines, AD and BE; they will cover the same distances in the same time. So, let AD and BE be equal distances, and from the eye, Z, let the rays fall, ZA, ZD, ZB, and ZE. Now, since the angle AZD is less than the angle BZE, therefore the distance AD will

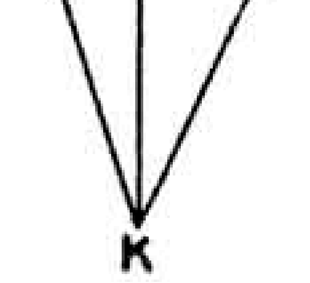


FIG. 51.

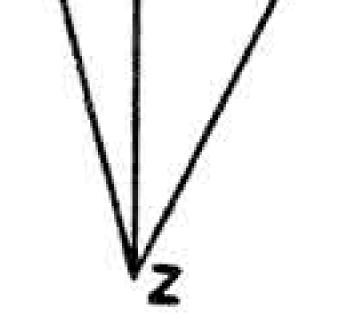


FIG. 52.

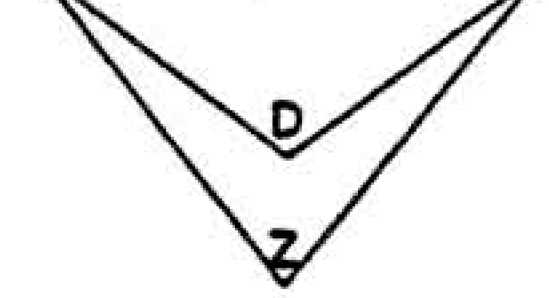


FIG. 53.

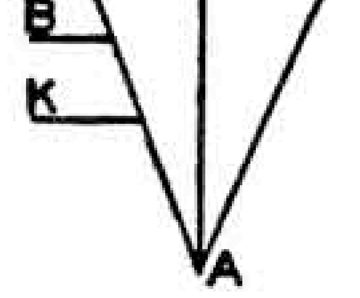


FIG. 54.

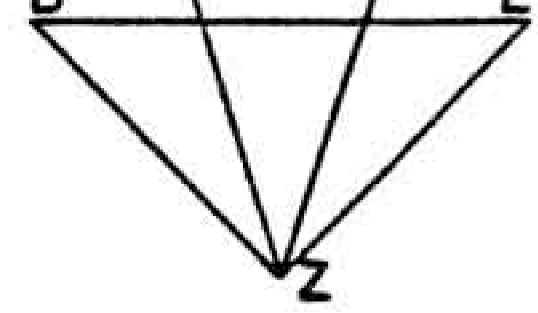


FIG. 54*a*.

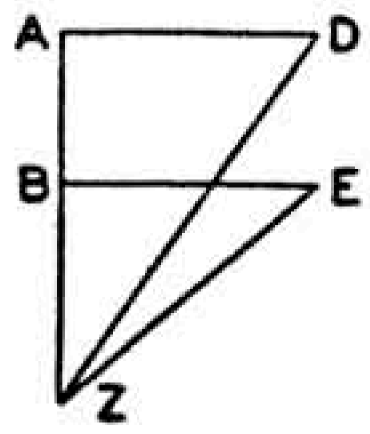


FIG. 54b.

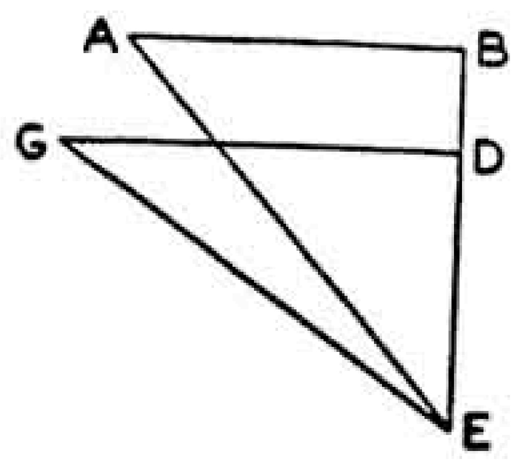


FIG. 55.

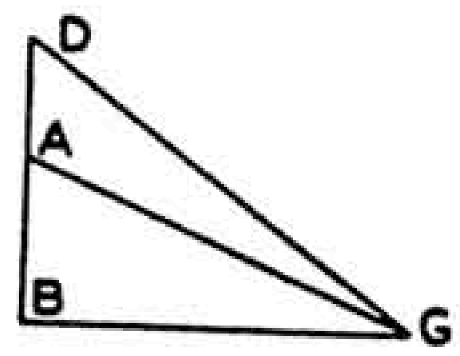


FIG. 56.

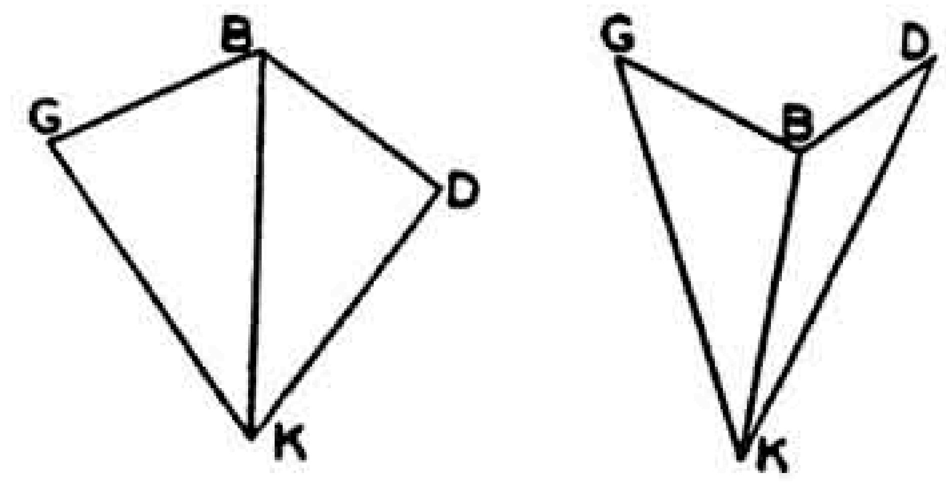


FIG. 57.

appear less than the distance BE. So that A will appear to move more slowly.

If the eye remains at rest, while the things seen are moved, the more remote of the things seen will seem to be left behind. (Fig. 55.)

Let A and G be the things seen, located on the straight lines, AB and GD, and let E be the eye, from which let the rays fall, EG, ED, EA, and EB. I say that the object at A will seem to have been left behind. Let ED be extended to a point where it meets AB, and let it be EB. Now, since the angle GEB is greater than the angle AEB, the distance GD appears greater than AB. So that if the eye remains at E, the rays, moving in the direction of A and G, will pass A more quickly than G. So AB will seem to be left behind.

Objects increased in size will seem to approach the eye. (Fig. 56.)

Let AB represent the size of the object seen, and let G be the eye, from which let the rays fall, GA and GB. And let BA be increased, and let it be BD, and let the ray fall, GD. Now, since the angle BGD is greater than the angle BGA, BD appears greater than BA. But things thought to be greater than themselves seem to be increased, and the things nearer the eye appear greater. So objects increased in size will seem to approach the eye.

When things lie at the same distance and the edges are not in line with the middle, it makes the whole figure sometimes concave, sometimes convex. (Fig. 57.)

For let GBD be seen by the eye located at K, and let the rays fall, KG, KB, and KD. So the whole figure will seem to be concave. Now let the thing seen in the middle be moved back, and let it be nearer the eye. So DBG will seem to be convex.

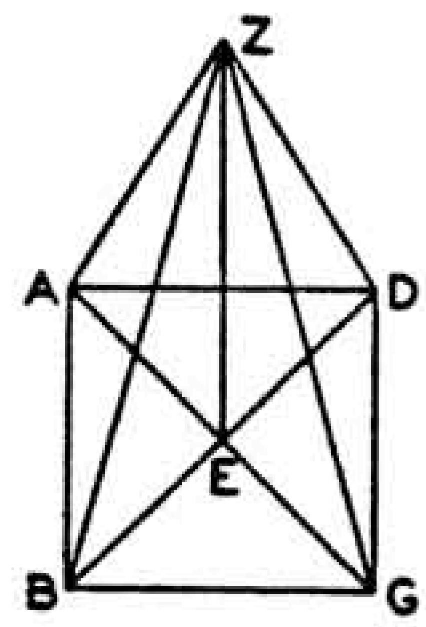


FIG. 58.

If from the meeting-point of the diameters of a square a perpendicular is drawn, and the eye rests upon this, the sides of the square will appear equal and the diameters also will appear equal. (Fig. 58.)

Let $ABGD$ be a square and let the diagonals DB and GA be drawn, and from E let the straight line EZ be drawn perpendicular to the plane, and upon this line let Z indicate the position of the eye, and let the rays fall, ZA, ZB, ZG, and ZD. Now, since DE is equal to EG, and EZ is common to them, and the angles are right angles, the base ZG is equal to the base DZ, and of the angles at the bases those are equal upon which equal sides subtend. So the angle EZG is equal to the angle EZD. Therefore EG will appear equal to ED. Similarly also the angle AZE is equal to the angle BZE. Therefore AG will appear equal to BD. Again, since GZ is equal to ZB, but also AB to GD, the three are equal to the three others, and one angle is equal to another angle. So one side will appear equal to the other, as also the remaining sides will appear equal.

But if a line is drawn from the eye to the meeting-point of the diameters, and if it is not perpendicular to the plane or equal to each one of the lines drawn from the meeting-point to the angles of the square, and if it does not make equal angles with them, the diameters will appear unequal. For similarly we shall prove what happens, just as in the case of circles.