COLOR SPACE

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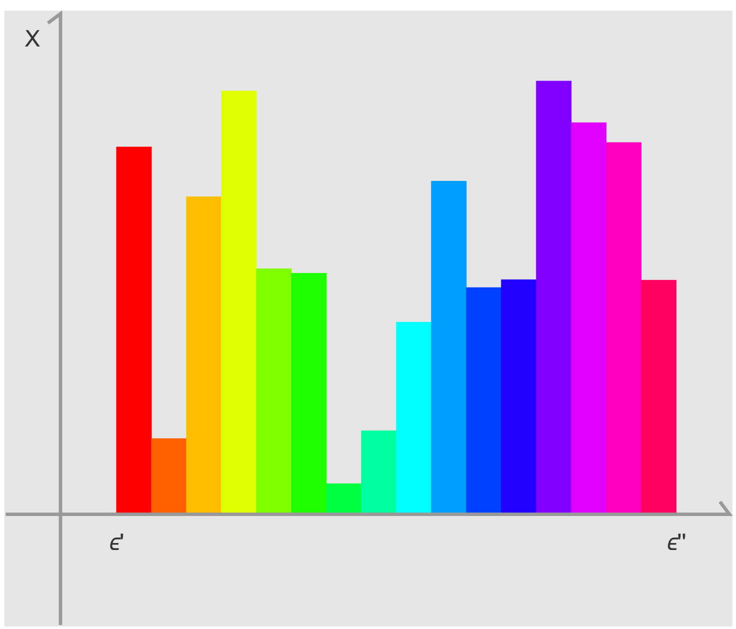
Introduction

01° We invest no effort in defining Light or Color. Rather, we represent the family $\mathcal{L}$ of Physical Lights and the corresponding family $\mathbf{\Lambda}$ of Perceptual Lights in terms of abstractions, guided by simple observations. The merits lie in the fruits. We will refer to the family of perceptual lights as Color Space. Step by step, we will supply Color Space with basic structures, in particular, the Linear and the Metric. In the end, we will draw connections, as appropriate, between Color Space and the neurophysiological processes of Color Perception. In this way, we will come to understand the members of Color Space as *colors*.

Abstract Color Space

02° We suggest physical lights by histograms, of the following form:



Physical Light

Such a histogram represents a physical light in its simplest form: as a Beam. The beam is a legion of *visible* photons traveling along mutually parallel straight line paths. The corresponding energies may vary from photon to photon but altogether they are bounded, below by ϵ' and above by ϵ'':

$$\epsilon' \sim 300 \; zeptojoules \; , \quad \epsilon'' \sim 500 \; zeptojoules$$

03° Of course, we may refer to the direction of the beam, without ambiguity. For now, let us assume that all beams travel in the same direction.

04° We describe the intensity X (that is, the joules per second) of the beam, as follows. We introduce a partition of the interval formed by the bounds, marking n (closely) spaced energies between ϵ' and ϵ'':

$$\epsilon' = \epsilon_0 < \epsilon_1 < \; \cdots \; < \epsilon_j < \; \cdots \; < \epsilon_n = \epsilon''$$

In turn, we introduce a (transparent) plane, perpendicular to the direction of the beam. For any time t and for any index j ($1 \leq j \leq n$), we observe the photons ϕ which pass through the plane in one second, but just those for which:

$$\epsilon_j < \epsilon_\phi \leq \epsilon_{j+1}$$

We measure the sum:

$$0 \leq X^j$$

of the energies of the photons so observed. In this way, we obtain (at least in principle) the intensity X of the beam at time t, graded by energy:

$$X = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^j \\ \vdots \\ X^n \end{pmatrix}$$

The components X^j of X would be measured in watts, that is, in joules per second.

05° Summing over the gradation, we obtain the (total) intensity of the beam:

$$\iota(X) = \sum_{j=1}^{n} X^j$$

Of course, $0 \leq \iota(X)$.

06° Very likely, the measurements just described, for physical lights and for histograms, will prove to depend upon time. But, for present purposes, let us assume that the (graded) intensity of the beam is constant in time.

07° Now let us agree to identify (abstract) physical lights Ξ by the sequences X, as displayed in article 4°. In effect, the components of X serve as (specific, very natural) coordinates for Ξ. We will require no others. We will refer only to X and say no more of Ξ.

08° Anticipating further developments, we appeal to Humpty Dumpty, who once said, in a scornful tone:

> "When I use a word, it means just what I choose it to mean – nothing more nor less."

09° Let us say that two physical lights X and Y in $\mathcal{L}$ are *equivalent* iff, should both fall upon a screen of perfect reflectance, a Standard Observer would perceive them to be indistinguishable. We will express this relation by writing:

$$X \sim Y$$

Informally, one may say that the physical lights X and Y are equivalent iff they determine the same perceptual light.

10° By common sense and by experience, we find that, for any physical lights X, Y, and Z in $\mathcal{L}$:

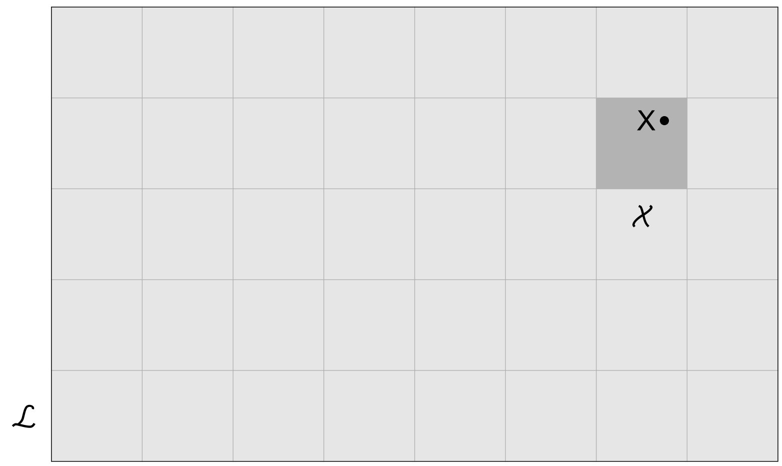
(1) $X \sim X$
(2) $X \sim Y \implies Y \sim X$
(3) $X \sim Y,\ Y \sim Z \implies X \sim Z$

Consequently, we may introduce certain subsets of $\mathcal{L}$, namely, those which consist of complete sets of mutually equivalent physical lights, and we may regard such subsets, in themselves, as perceptual lights. In particular, for each physical light X, we may introduce the perceptual light $\mathcal{X}$ to which it belongs:

$$X \in \mathcal{X}$$

One may say that the physical light X determines the perceptual light $\mathcal{X}$.

11° Now let us agree to identify perceptual lights with the sets $\mathcal{X}$, as just described, and let us denote by $\mathbf{\Lambda}$ the family of all such lights. We are committed to refer to $\mathbf{\Lambda}$ as Color Space. At this point, however, we have no specific coordinate system to offer.



Perceptual Light

12° Finally, let us draw from the foregoing abstractions the natural mapping μ carrying the family $\mathcal{L}$ of physical lights to the family $\mathbf{\Lambda}$ of perceptual lights, that is, to Color Space, as follows:

$$\mu(X) = \mathcal{X}$$

where X is any physical light and where $\mathcal{X}$ is the perceptual light to which X belongs. By this mapping, we shall express relations between basic structures on $\mathcal{L}$, defined in practice by objective measurement, and corresponding basic structures on $\mathbf{\Lambda}$, derived in practice by subjective judgement.

13° For physical lights X in $\mathcal{L}$, we have intensity $\iota(X)$, measured objectively in watts. For perceptual lights $\mathcal{X}$ in $\mathbf{\Lambda}$, we will, in due course, encounter luminosity $\lambda(\mathcal{X})$, measured subjectively by our Standard Observer in lumens. These measures of the "amounts" of light in X and $\mathcal{X}$, respectively, play fundamental roles in the study of Color Space.

Linear Color Space

14° For the family $\mathcal{L}$ of all physical lights, we introduce two operations, Addition and Scalar Multiplication. For any physical lights X, Y, and Z in $\mathcal{L}$ and for any nonnegative (pure) number c, we define:

$$X + Y, \quad cZ$$

as follows:

$$(X+Y)^j = X^j + Y^j, \quad (cZ)^j = cZ^j$$

where, of course, j is any one of the relevant indices: $1 \leq j \leq n$.

15° The intensity of the physical light $X+Y$ on the interval $(\epsilon_{j-1}, \epsilon_j\,]$ is the sum of the intensities of the physical lights X and Y, separately, on that interval. The intensity of the physical light cZ on the interval $(\epsilon_{j-1}, \epsilon_j\,]$ is the product of c and the intensity of the physical light Z on that interval.

16° The foregoing operations have many properties, which devolve from the familiar properties of ordinary arithmetic. For instance, $c(X+Y) = cX + cY$. We need not list them all, but we will make use of them.

17° For any physical lights X, Y, and Z and for any nonnegative (pure) number c, we find, by the extensive experience of our Standard Observer:

(4) $X \sim Y \implies X + Z \sim Y + Z$

(5) $X \sim Y \implies cX \sim cY$

One may justifiably argue that the foregoing implications provide the basic connections between physical and perceptual lights. In any case, they provide support for transporting the operations of addition and scalar multiplication from $\mathcal{L}$ to $\mathbf{\Lambda}$, as follows.

18° For any perceptual lights $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ in $\mathbf{\Lambda}$ and for any nonnegative (pure) number c, we define:

$$\mathcal{X} + \mathcal{Y}, \quad c\,\mathcal{Z}$$

by the following maneuvers. We select, in manner arbitrary, physical lights X' and X'' in $\mathcal{X}$, Y' and Y'' in $\mathcal{Y}$, and Z' and Z'' in $\mathcal{Z}$. By implications (4) and (5), we find that:

$$X' + Y' \sim X'' + Y' \sim X'' + Y''$$

and:

$$cZ' \sim cZ''$$

Now $X' + Y'$ and $X'' + Y''$ determine the same perceptual light and cZ' and cZ'' determine the same perceptual light. Naturally, we define the former to be $\mathcal{X} + \mathcal{Y}$ and the latter to be $c\,\mathcal{Z}$. In this way, we transport the operations of addition and scalar multiplication from $\mathcal{L}$ to $\mathbf{\Lambda}$.

19° The natural mapping μ carrying $\mathcal{L}$ to $\mathbf{\Lambda}$ provides a neat summary of the foregoing matter. For any physical lights X, Y, and Z in $\mathcal{L}$ and for any nonnegative (pure) number c:

$$\mu(X+Y) = \mu(X) + \mu(Y), \quad \mu(cZ) = c\mu(Z)$$

20° The families $\mathcal{L}$ and $\mathbf{\Lambda}$ of physical lights and perceptual lights both carry the structure of a *cone*. That is, one can add lights and one can multiply them by nonnegative (pure) numbers, subject to familiar properties. Moreover, the cones are *nondegenerate*. That is, one can identify a light, let it be Θ, such that, for any lights L' and L'', if $L' + L'' = \Theta$ then $L' = \Theta$ and Ł$'' = \Theta$. One refers to Θ as the *vertex* of the cone. For $\mathcal{L}$, Θ would be O. For $\mathbf{\Lambda}$, Θ would be the *imperceptible* light $\mu(O)$:

$$O = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mu(O) = \emptyset$$

The components of O carry units of (zero) watts.

21° Intensity is linear on $\mathcal{L}$. That is, for any physical lights X, Y, and Z and for any nonnegative (pure) number c:

$$\iota(X+Y) = \iota(X) + \iota(Y), \qquad \iota(cZ) = c\iota(Z)$$

One may expect (hope (?)) that the same is true for luminosity on $\mathbf{\Lambda}$. That is, for any perceptual lights $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ and for any nonnegative (pure) number c:

$$\lambda(\mathcal{X}+\mathcal{Y}) = \lambda(\mathcal{X}) + \lambda(\mathcal{Y}), \qquad \lambda(c\mathcal{Z}) = c\lambda(\mathcal{Z})$$

In fact, it proves to be so. See article 28°.

22° We ought note that there is a third "measure" of the amount of light: the *brightness* $\beta(\mathcal{X})$ of a perceptual light $\mathcal{X}$. The relation between $\beta(\mathcal{X})$ and $\lambda(\mathcal{X})$ is subtle: indeed, not linear but logarithmic. It falls under a general study by Gustav Fechner of the relation between "response" and "stimulus." We will describe it later, very briefly.

23° For the family $\mathcal{L}$ of physical lights, we find a *natural basis* of lights having minimal structure, in terms of which all physical lights can be uniquely defined. We will call these lights *spectral* lights:

$$\Delta_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \Delta_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \Delta_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

In sharp terms:

$$\Delta_k^j = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

where $1 \leq j \leq n$ and $1 \leq k \leq n$.

24° By design, the energies of the photons in the spectral light Δ_k all lie in the interval $(\epsilon_{k-1}, \epsilon_k\,]$ and the sum of the energies per second equals one watt. Obviously, for any physical light X, we can represent X as a linear combination of spectral lights:

$$X = \sum_{k=1}^{n} \hat{X}^k \Delta_k$$

The coefficients $\hat{X}^k$ are nonnegative pure (!) numbers, numerically equal to X^k. The representation is unique (!).

25° For the family $\mathbf{\Lambda}$ of perceptual lights, we find no such natural basis. However, by the mapping μ we can transport the spectral lights just described from $\mathcal{L}$ to $\mathbf{\Lambda}$:

$$\mathcal{D}_k = \mu(\Delta_k)$$

where k is any relevant index: $1 \leq k \leq n$. We will call these (perceptual) lights *simple* lights. Our Standard Observer reports that spectral lights are distinguishable, hence that simple lights are distinct.

26° Then, following systematic comparisons, the Observer reports certain constants, as follows. By subjective (but careful) comparisons, he determines the index j such that, for all indices k $(1 \leq k \leq n)$, the simple light $\mathcal{D}_j$ is at least as "bright" as the simple light $\mathcal{D}_k$. He sets the value of V_j equal to 1. In turn, for each index k $(1 \leq k \leq n)$, he finds the positive number V_k

$(0 < V_k \leq 1)$ such that $V_k \mathcal{D}_j$ and $\mathcal{D}_k$ are of equal brightness. He presents his report:

$$\lambda(\mathcal{D}_k) = V_k \, \iota(\Delta_k) \qquad\qquad (\iota(\Delta_k) = 1,\ 0 < V_k \leq V_j = 1))$$

The constants carry the units of lumens per watt.

27° Let $\mathcal{X}$ be a perceptual light. Let X be any physical light which determines $\mathcal{X}$. By article 19°, we find that:

$$\begin{aligned}
\mathcal{X} &= \mu(X) \\
&= \mu(\sum_{k=1}^{n} \hat{X}^k \Delta_k) \\
&= \sum_{k=1}^{n} \hat{X}^k \mu(\Delta_k) \\
&= \sum_{k=1}^{n} \hat{X}^k \mathcal{D}_k
\end{aligned}$$

Consequently, every perceptual light can be expressed as a linear combination of simple lights. However, the representation is far (!) from unique. In any case, we find the following important inference:

$$\begin{aligned}
\lambda(\mathcal{X}) &= \sum_{k=1}^{n} \hat{X}^k \lambda(\mathcal{D}_k) \\
&= \sum_{k=1}^{n} \hat{X}^k V_k \, \iota(\Delta_k) \\
&= \sum_{k=1}^{n} V_k \, \iota(\hat{X}^k \Delta_k) \\
&= \sum_{k=1}^{n} V_k X^k
\end{aligned}$$

28° Let us call attention to certain basic lights, the *white* lights:

$$W = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \mathcal{W} = \mu(W)$$

We find:

$$\lambda(\mathcal{W}) = \omega \qquad (\omega = \sum_{k=1}^{n} V_k)$$

29° At this point, we are prepared to call upon the Fundamental Principle of Color Theory:

> **COLOR SPACE IS A 3-DIMENSIONAL NONDEGENERATE CONE**

The principle is based upon extensive experiments by Observers, involving the comparison of various linear combinations of perceptual lights.

30° In precise terms, the principle means that there exist perceptual lights $\mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_3$ in $\mathbf{\Lambda}$ such that, for every perceptual light $\mathcal{X}$ in $\mathbf{\Lambda}$, there exist (pure) numbers c^1, c^2, and c^3 such that:

$$\mathcal{X} = c^1\mathcal{Z}_1 + c^2\mathcal{Z}_2 + c^3\mathcal{Z}_3$$

In this context, the numbers c^1, c^2, and c^3 are unique. However, one or two of the numbers may be negative, in which case one reinterprets the equality by shifting the corresponding terms "to the other side." After all, we can form constant multiples of a perceptual light only if the constant is nonnegative (and in fact pure).

31° That said, we have:

$$\lambda(\mathcal{X}) = c^1\lambda(\mathcal{Z}_1) + c^2\lambda(\mathcal{Z}_2) + c^3\lambda(\mathcal{Z}_3)$$

even if one or two of the coefficients are negative. For instance, if $c^1 < 0$ while $0 \leq c^2$ and $0 \leq c^3$ then:

$$(-c^1)\mathcal{Z}_1 + \mathcal{X} = c^2\mathcal{Z}_2 + c^3 + \mathcal{Z}_3$$

so that:

$$(-c^1)\lambda(\mathcal{Z}_1) + \lambda(\mathcal{X}) = c^2\lambda(\mathcal{Z}_2) + c^3\lambda(\mathcal{Z}_3)$$

We need say no more.

32° Given the relations in article 31°, one refers to $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$ as a basis of *primary* perceptual lights for $\mathbf{\Lambda}$. However, at this point, one such basis is as good as any other.

33° Now let us invoke the algebraic and analytic methods of Cartesian Geometry to develop a sharp interpretation of the foregoing principle. We introduce the cartesian spaces:

$$\mathbf{R}^3, \quad \mathbf{R}^n$$

The elements of these spaces stand as follows:

$$K = \begin{pmatrix} K^1 \\ K^2 \\ K^3 \end{pmatrix}, \qquad X = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^j \\ \vdots \\ X^n \end{pmatrix}$$

The entries are numbers, unconstrained. For the former, the units are lumens. For the latter, watts. Of course, the spectral lights:

$$\Delta_1, \ \Delta_2, \ \dots \ , \Delta_j, \ \dots \ , \ \Delta_n$$

form a basis for $\mathbf{R}^n$, while the elements:

$$E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for $\mathbf{R}^3$.

34° In turn, we introduce the subsets:

$$\mathbf{K}^3, \quad \mathbf{K}^n$$

of $\mathbf{R}^3$ and $\mathbf{R}^n$, respectively. The entries are constrained to be nonnegative numbers. Clearly, $\mathbf{K}^3$ and $\mathbf{K}^n$ are nondegenerate cones. The vertices are the origins:

$$O = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

35° Obviously, $\mathcal{L} = \mathbf{K}^n$.

36° By the Fundamental Principle of Color Theory, we may (!) identify our Color Space $\mathbf{\Lambda}$ with a nondegenerate 3-dimensional cone in $\mathbf{R}^3$.

37° To do so, we introduce (in abstract terms) a basis $\mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_3$ of primary perceptual lights in $\mathbf{\Lambda}$. In turn, we choose (at liberty) a basis Γ_1, Γ_2, and Γ_3 for $\mathbf{R}^3$. Finally, we identify each perceptual light $\mathcal{X}$ in $\mathbf{\Lambda}$ with a corresponding element K of $\mathbf{R}^3$, as follows

$$\mathcal{X} = C^1\mathcal{Z}_1 + C^2\mathcal{Z}_2 + C^3\mathcal{Z}_3 \quad \Longleftrightarrow \quad K = C^1\Gamma_1 + C^2\Gamma_2 + C^3\Gamma_3$$

Of course, the primary lights $\mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_3$ correspond to the elements Γ_1, Γ_2, and Γ_3, respectively. The coefficients C^1, C^2, and C^3 may be any nonnegative (pure) numbers whatsoever, but, as usual, some among them may be negative.

38° In any case, as $\mathcal{X}$ runs through $\mathbf{\Lambda}$, so K runs through a cone, let it be $\mathbf{L}$, in $\mathbf{R}^3$.

39° The preceding maneuvers serve to supply the abstract space $\mathbf{\Lambda}$ with coordinates. In practice, special choices prove useful. For instance, we might insist that:

$$\mathbf{K}^3 \subseteq \mathbf{L}$$

We might just as well insist that:

$$\Gamma_1 = E_1,\ \Gamma_2 = E_2,\ \Gamma_3 = E_3$$

and that:

$$\lambda(\mathcal{Z}_1) = 1,\ \lambda(\mathcal{Z}_2) = 1,\ \lambda(\mathcal{Z}_3) = 1$$

We will refer to such a choice as an *empirical* choice.

40° But we might insist that:

$$\mathbf{L} \subseteq \mathbf{K}^3$$

while

$$\omega\mathcal{Z}_1 + \omega\mathcal{Z}_2 + \omega\mathcal{Z}_3 = \mathcal{W}$$

and:

$$\lambda(\mathcal{Z}_1) = \frac{1}{3},\ \lambda(\mathcal{Z}_2) = \frac{1}{3},\ \lambda(\mathcal{Z}_3) = \frac{1}{3}$$

We will refer to such a choice as a *theoretical* choice.

41° in 1931, the International Commission on Illumination promoted a particular theoretical choice. It serves now as a base for all coordinate computations, without recourse to further experimentation.

42° Now we may represent the linear (!) mapping μ by the matrix:

$$M = \begin{pmatrix} M_1^1 & M_2^1 & \cdots & M_k^1 & \cdots & M_n^1 \\ M_1^2 & M_2^2 & \cdots & M_k^2 & \cdots & M_n^2 \\ M_1^3 & M_2^3 & \cdots & M_k^3 & \cdots & M_n^3 \end{pmatrix}$$

In terms of coordinates:

$$\mathcal{X} = C^1\mathcal{Z}_1 + C^2\mathcal{Z}_2 + C^3\mathcal{Z}_3 \quad \Longleftrightarrow \quad K = C^1\Gamma_1 + C^2\Gamma_2 + C^3\Gamma_3$$

the relation stands as follows::

$$(m) \qquad \begin{pmatrix} M_1^1 & M_2^1 & \cdots & M_k^1 & \cdots & M_n^1 \\ M_1^2 & M_2^2 & \cdots & M_k^2 & \cdots & M_n^2 \\ M_1^3 & M_2^3 & \cdots & M_k^3 & \cdots & M_n^3 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^k \\ \vdots \\ X^n \end{pmatrix} = \begin{pmatrix} K^1 \\ K^2 \\ M^3 \end{pmatrix}$$

Clearly, the entries M_k^j in M must be measured in lumens per watt.

43° With reference to article 27°, we find that, for each index k $(1 \leq k \leq n)$:

$$\mathcal{D}_k = \mu(\Delta_k) = \iota(\Delta_k) \begin{pmatrix} M_k^1 \\ M_k^2 \\ M_k^3 \end{pmatrix} = \hat{M}_k^1\Gamma_1 + \hat{M}_k^2\Gamma_2 + \hat{M}_k^3\Gamma_3$$

Hence:

$$\lambda(\mathcal{D}_k) = M_k^1 + M_k^2 + M_k^3$$

44° Let us emphasize the fundamental empirical fact which underlies our theoretical structure. For any index k $(1 \leq k \leq n)$:

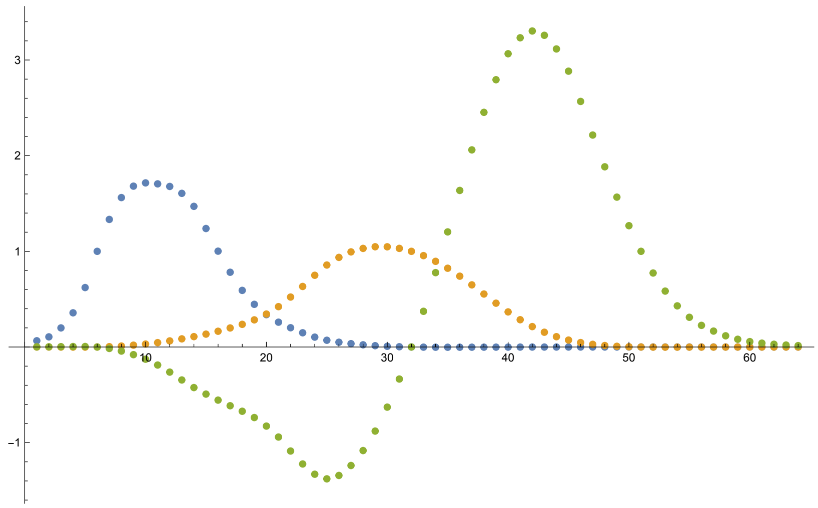
$$\hat{M}_k^1, \;\; \hat{M}_k^2, \;\; \hat{M}_k^3$$

are the numbers of lumens of the primary perceptual lights Γ_1, Γ_2, and Γ_3, respectively, which figure in the representation of the simple light $\mathcal{D}_k$. One usually finds the rows of M displayed in three tables, or graphs, to represent the results of the basic matching experiments.

45° For instance, following article 37°, we may make selections for our primary perceptual lights:

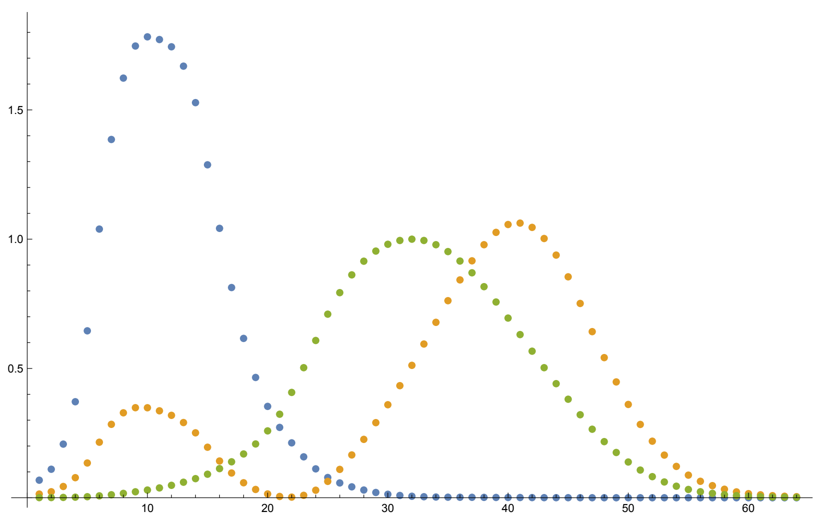
$$\mathcal{Z}_1 = \frac{1}{V_{06}}\mathcal{D}_{06}, \quad \mathcal{Z}_2 = \frac{1}{V_{32}}\mathcal{D}_{32}, \quad \mathcal{Z}_3 = \frac{1}{V_{51}}\mathcal{D}_{51}$$

They are (within factors) simple lights. The corresponding graphs take the following form.

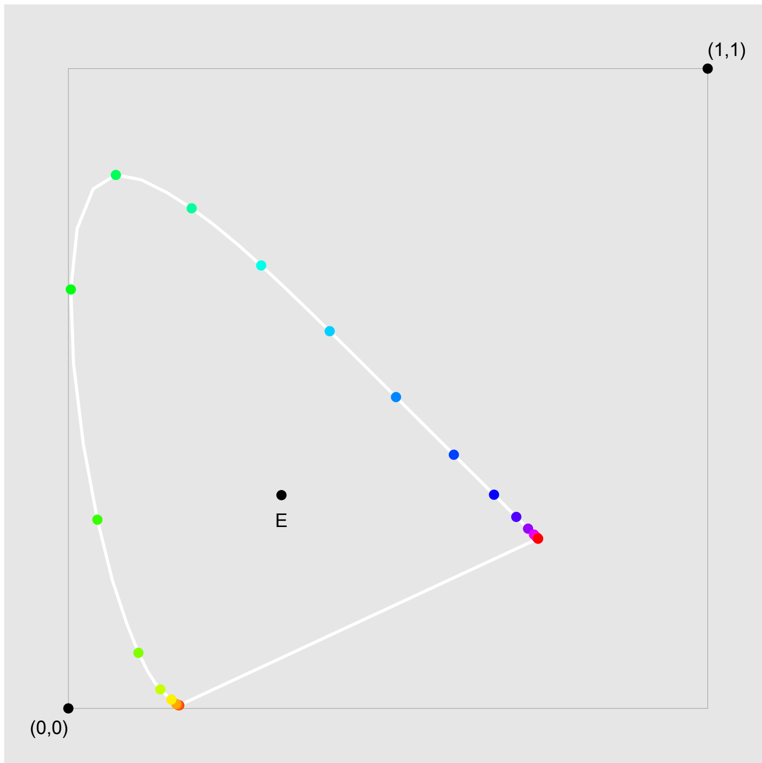


TriStimulus Values: Practical

46° The foregoing discussion follows a "practical" path. In due course, we will describe the over arching "theoretical" path, which for computation includes all other paths.



TriStimulus Values: Theoretical



Chromaticity