Mathematical Appendixes

APPENDIX A
POWERS-OF-TEN NOTATION

Large numbers crop up as soon as you talk quantitatively about light waves; the largest frequency we will mention will be a hundred billion billion times as large as the smallest, and the smallest wavelength will be about one ten-thousandth of a billionth of an inch. We would like to find a way to avoid these rather cumbersome forms of numbers.

An extremely useful and fairly simple shorthand for dealing with numbers that span a large range is the ***powers-of-ten notation.*** Instead of writing:

one hundred billion billion
= 100,000,000,000,000,000,000 (20 zeros)

we write:

10^{20} (which means, a 1 followed by 20 zeros)

How do we get this? It is just a familiar rule of raising a number to a power. For example, consider:

one thousand = 1000 = 10^3

according to the notation we just introduced. By the usual rule of what an exponent means, 10^3 is the "cube of ten," which gives the same result:

$$10^3 = 10 \times 10 \times 10 = 1000$$

So 10 raised to any power is 10 multiplied by itself that many times, which is 1 with that many zeros after it.

What about 1 itself—that is, a 1 with no zeros after it? In our notation we must write this as:

$$1 = 10^0$$

How about small numbers, say 0.1? Instead of multiplying 1 by powers of 10, this requires dividing 1 by powers of 10. The number of 10's we divide by will be indicated by negative powers, for example

one tenth = 0.1 = $\frac{1}{10}$ = 10^{-1}

or:

one ten thousandth of a billionth
= 1/10,000,000,000,000
= 10^{-13} (count the zeros!).

In addition to brevity, this notation has another advantage: multiplication and division of such "powers-of-ten" numbers is easy—you need only add or subtract the exponents. For example, $10^{13} \times 10^4$ is 13 tens times 4 tens all multiplied together, so the product is (13 + 4) tens, 17 tens, or 10^{17}:

$$10^3 \times 10^4 = 10^{13+4} = 10^{17}$$

Similarly $10^{-2} \times 10^{-1}$ is

$$\frac{1}{10 \times 10} \times \frac{1}{10} = \frac{1}{(10 \times 10) \times 10},$$

or:

$$10^{-2} \times 10^{-1} = 10^{-2+(-1)} = 10^{-3}$$

Thus multiplication becomes simple addition. Division is just as easy when we note, for example $1/10^{13} = 10^{-13}$, so dividing by 10 to a power is the same as multiplying by 10 to the negative power. Thus,

$$\frac{10^{13}}{10^{13}} = 10^{13} \times 10^{-13} = 10^0$$
$$= 1 \text{ (of course)}$$

APPENDIX B

THE MATHEMATICAL FORM OF SNELL'S LAW

To state the mathematical form of Snell's law, we shall first need the trigonometric function called the *sine* of an angle θ, written as sin θ. To do this, we construct a right triangle with one angle equal to θ (Fig. B.1a). Then sin θ is equal to the length of the side opposite θ, divided by the length of the hypotenuse:

$$\sin\theta = \frac{p}{r} \qquad \text{(B1)}$$

For any value of the angle θ, this function can be found in tables or calculators.

Now, consider a beam of light incident on a boundary at angle θ_i, in material of index of refraction n_i, and then transmitted at angle θ_t, in material of index of refraction n_t (Fig. B.1b). We have drawn the wavefront AA', which is perpendicular to the incident beam. Hence (since the surface AB' is perpendicular to its own normal) the angle $\sphericalangle A'AB'$ must equal θ_i. Then:

$$\sin\theta_i = \frac{\overline{A'B'}}{\overline{AB'}}$$

or:

$$\frac{1}{\overline{AB'}} = \frac{1}{\overline{A'B'}}\sin\theta_i \qquad \text{(B2)}$$

Drawing the wavefront BB', perpendicular to the transmitted beam, and going through the same steps, gives us:

$$\frac{1}{\overline{AB'}} = \frac{1}{\overline{AB}}\sin\theta_t \qquad \text{(B3)}$$

Comparing Equations B2 and B3 gives us:

$$\frac{1}{\overline{A'B'}}\sin\theta_i = \frac{1}{\overline{AB}}\sin\theta_t \qquad \text{(B4)}$$

Suppose that it took light a time T to travel between the two wave fronts, that is, from A' to B' for the part still on the incident side, and from A to B for the part on the transmitted side. If the light travels on the incident side with speed $v_i = c/n_i$, and on the transmitted side with speed $v_t = c/n_t$, then we have:

$$\overline{A'B'} = v_iT = \frac{cT}{n_i}, \quad \text{and}$$

$$\overline{AB} = v_tT = \frac{cT}{n_t}$$

Putting these results in Equation B4 gives us:

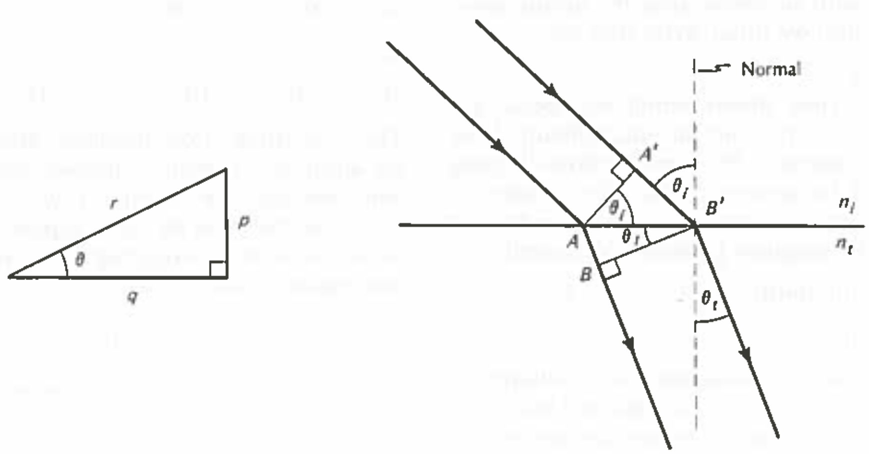
$$\frac{n_i}{cT}\sin\theta_i = \frac{n_t}{cT}\sin\theta_t$$

or, *Snell's law:*

$$\mathbf{n_i \sin\theta_i = n_t \sin\theta_t} \qquad \text{(B5)}$$

FIGURE B.1

(a) Right triangle with one angle equal to *θ*. (b) Light incident from a fast medium to a slow medium ($n_i < n_t$) at angle θ_i. The refracted beam bends toward the normal; that is, $\theta_t < \theta_i$. Two wavefronts, *AA'* and *BB'* are shown.



(a) (b)

If we refer back to Figure B.1a, we notice that p can never be larger than r—the hypotenuse is always the largest side. Hence, from Equation B1, sin θ can never be greater than 1. When $\sin\theta_t$ equals 1, we have the condition of *total internal reflection*. That is, the incident angle is then the *critical angle* θ_c, where (from Eq. B5, setting $\sin\theta_t = 1$):

$$\mathbf{n_i \sin\theta_c = n_t} \qquad \text{(B6)}$$

Any incident angle greater than θ_c will result in a totally internally reflected light beam. There is then no transmitted beam—Equation B5 cannot be satisfied. Since $\sin\theta_c$ must also be less than 1, we can only satisfy Equation B6 if n_i is greater than n_t. That is, only when going from a slower medium (say glass) toward a faster medium (say air) can light be totally internally reflected.

APPENDIX C

THE FOCAL POINT OF A CONVEX MIRROR

To prove the relation given in Section 3.3A, that the focal length is one half the radius of a spherical convex mirror: $f = \overline{OF} = \frac{1}{2}\overline{OC}$, consider Figure C.1. An incident ray,

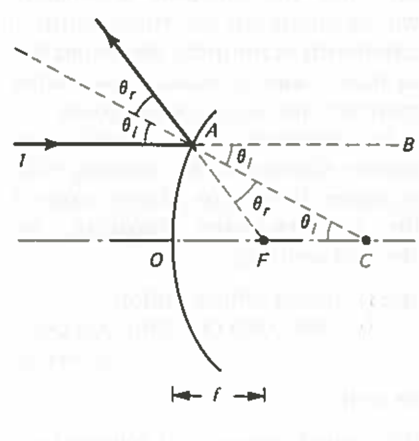


FIGURE C.1

Ray *1*, traveling parallel to the axis, strikes a convex mirror at *A*, and leaves as if it had come from the focal point, *F*.

ray *1*, traveling parallel to the axis, strikes the mirror at *A*, with angle of incidence θ_i. It is reflected with angle of reflection θ_r, as shown. Since *AB* is a continuation of the incident ray, and *AC* is the normal at *A*, the angle $\angle CAB$ is equal to θ_i. The angle $\angle ACF$ must then also be equal to θ_i, because it and $\angle CAB$ are opposite interior angles between two parallel lines. Similarly, *AF* is a continuation of the reflected ray, so the angle $\angle CAF$ must equal θ_r, which by the law of reflection is equal to θ_i. Hence the triangle $\triangle CAF$ is an isosceles triangle, having two equal angles ($\angle CAF$ and $\angle ACF$), and therefore two equal sides

$$\overline{AF} = \overline{FC}$$

Now, for *paraxial* rays, the point *A* must be close to the point *O*, so $\overline{AF} = \overline{OF}$. Hence, we can write:

$$\overline{OF} = \overline{FC} \text{ (paraxial rays)}$$

But since $\overline{OC} = \overline{OF} + \overline{FC}$, we have, finally:

$$\overline{OF} = \tfrac{1}{2}\,\overline{OC}$$

That is, the focal length of the spherical mirror is half its radius.

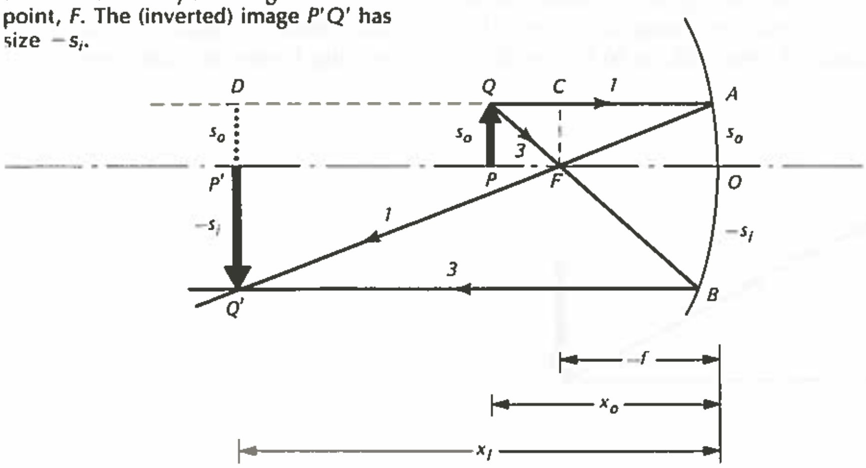


FIGURE D.1

Object *PQ* of size s_o emits ray *1* parallel to the axis and ray *3* through the focal point, *F*. The (inverted) image *P'Q'* has size $-s_i$.

APPENDIX D
THE MIRROR EQUATION

To derive the ***mirror equation***, we redraw Figure 3.17, showing only rays *1* and *3*, but extending ray *1* backward (Fig. D.1). We then note pairs of *similar* triangles (treating *AOB* as a straight line): $\triangle ABQ$ and $\triangle CFQ$, also $\triangle CFA$ and $\triangle DQ'A$. From the first pair of similar triangles, we may write the equal ratios:

$$\frac{\overline{CF}}{\overline{AB}} = \frac{\overline{QC}}{\overline{QA}}$$

We note that $\overline{CF} = s_o$, the object size; $\overline{AB} = s_o - s_i$, where $-s_i$ is the image size (the minus sign indicates that the image points *downward*); $\overline{QA} = x_o$, the distance of the object in front of the mirror; and $\overline{QC} = x_o - (-f) = x_o + f$, where f is the focal length of the mirror and is negative for this concave mirror. Hence the above equation reads:

$$\frac{s_o}{s_o - s_i} = \frac{x_o + f}{x_o} = 1 + \frac{f}{x_o} \qquad \text{(D1)}$$

The second pair of similar triangles gives:

$$\frac{\overline{CF}}{\overline{DQ'}} = \frac{\overline{CA}}{\overline{DA}}$$

But $\overline{CF} = s_o$, $\overline{DQ'} = s_o - s_i$, $\overline{CA} = -f$, and $\overline{DA} = x_i$, the distance of the image in front of the mirror. So this equation reads:

$$\frac{s_o}{s_o - s_i} = \frac{-f}{x_i} \qquad \text{(D2)}$$

Since the left-hand sides of Equations D1 and D2 are the same, their right-hand sides must be equal:

$$\frac{-f}{x_i} = 1 + \frac{f}{x_o}, \quad \text{or} \quad -\frac{f}{x_o} - \frac{f}{x_i} = 1$$

Dividing both sides of the latter form of the equation by f then gives:

$$-\frac{1}{x_o} - \frac{1}{x_i} = \frac{1}{f} \qquad \text{(D3)}$$

This is the *mirror equation*, derived here for the case of the *concave* mirror, where f is *negative*. It also is valid for *convex* mirrors: you just put in a *positive* f. If you should find x_i to be *negative*, it simply means that the image is that distance *behind* the mirror (a *virtual* image).

Equation D3 locates the image; what about the size of the image? If we write Equation D2 upside down, we get:

$$-\frac{x_i}{f} = \frac{s_o - s_i}{s_o} = 1 - \frac{s_i}{s_o}$$

or:

$$\frac{s_i}{s_o} = 1 + \frac{x_i}{f} \qquad \text{(D4)}$$

Hence, if we know the location of the image x_i, then Equation D4 tells us its size s_i. We can simplify this by using the mirror equation, Equation D3. Multiplying it by x_i gives:

$$-\frac{x_i}{x_o} = 1 + \frac{x_i}{f}$$

Using this in Equation D4 gives:

$$\frac{s_i}{s_o} = -\frac{x_i}{x_o} \qquad \text{(D5)}$$

This equation tells us how big the image is, compared to the object—how much it is *magnified*. The minus sign tells us that if x_i and x_o are both positive (in front of the mirror, as in Fig. D.1), then s_i will have the opposite sign to s_o—it will be *inverted*, as in Figure D.1.

Thus, Equations D3 and D4 tell us the size, position, and orientation of the image.

APPENDIX E

THE LENS EQUATION

To derive the ***lens equation,*** we redraw Figure 3.26, showing only rays *1* and *3*, and use the technique of Appendix D (Fig. E.1). The pairs of similar triangles here are $\triangle ABQ$ and $\triangle CFQ$, as well as $\triangle ABQ'$ and $\triangle AOF'$. The first pair gives:

$$\frac{\overline{CF}}{\overline{AB}} = \frac{\overline{QC}}{\overline{QA}}$$

Here $\overline{CF} = s_o$, the object size; $\overline{AB} = s_o - s_i$, where $-s_i$ is the image size (again, it has a minus sign because the image points *downward*); $\overline{QA} = x_o$, the distance of the object in *front* of the lens; and $\overline{QC} = x_o - f$, where f is the focal length of the lens. Hence the above equation reads:

$$\frac{s_o}{s_o - s_i} = \frac{x_o - f}{x_o} = 1 - \frac{f}{x_o}. \qquad \text{(E1)}$$

The second pair of similar triangles gives:

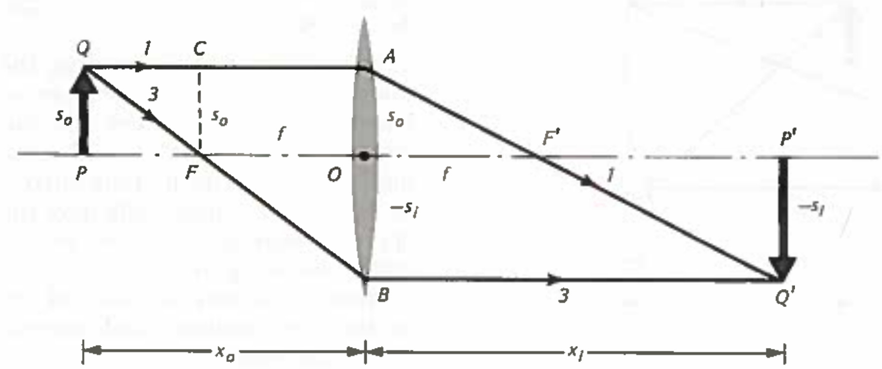
$$\frac{\overline{AO}}{\overline{AB}} = \frac{\overline{OF'}}{\overline{BQ'}}$$

But $\overline{AO} = s_o$, $\overline{AB} = s_o - s_i$, $\overline{OF'} = f$, and $\overline{BQ'} = x_i$, the distance of the image *past* the lens. So this equation reads:

$$\frac{s_o}{s_o - s_i} = \frac{f}{x_i} \qquad \text{(E2)}$$

FIGURE E.1

Object PQ of size s_o emits ray *1* parallel to the axis and ray *3* through the focal point, F. The (inverted) image $P'Q'$ has size $-s_i$.



Comparing Equations E1 and E2, we get:

$$\frac{f}{x_i} = 1 - \frac{f}{x_o}, \quad \text{or} \quad \frac{f}{x_o} + \frac{f}{x_i} = 1$$

Dividing both sides of the latter form of the equation by f then gives:

$$\frac{1}{x_o} + \frac{1}{x_i} = \frac{1}{f} \qquad \text{(E3)}$$

This is the *lens equation*, derived here for a *converging* lens, where f is *positive*. It also is valid for *diverging* lenses; you just put in a *negative* f. If you should find x_i to be *negative*, it simply means that the image is in *front* of the lens (a *virtual* image).

Equation E3 locates the image; what about the size of the image? As in Appendix D, we write Equation E2 upside down:

$$\frac{x_i}{f} = \frac{s_o - s_i}{s_o} = 1 - \frac{s_i}{s_o}$$

or:

$$\frac{s_i}{s_o} = 1 - \frac{x_i}{f} \qquad \text{(E4)}$$

Multiplying Equation E3 by x_i gives:

$$\frac{x_i}{x_o} + 1 = \frac{x_i}{f}, \quad \text{or} \quad 1 - \frac{x_i}{f} = -\frac{x_i}{x_o}$$

which, combined with Equation E4, gives:

$$\frac{s_i}{s_o} = -\frac{x_i}{x_o} \qquad \text{(E5)}$$

That is, as in the case of the mirror, the *magnification* is just the negative of the ratio of the image and object distances. Again, positive x_i and x_o (as in Fig. E.1) means there is a negative magnification—the image is *inverted*, as in Figure E.1.

APPENDIX F

TWO THIN LENSES TOUCHING

Consider two thin lenses with focal lengths f_1 and f_2. Start with them separated by a distance t, with an object at the first focal plane of the first lens. Light from that object must then emerge from the first lens in a parallel beam (Fig. F.1a). But a parallel beam incident on the second lens will be focused in that lens' second focal plane, as shown in the figure.

This is true no matter what the separation t is. If we let the separation vanish, so the two lenses touch, we can think of them as one *effective* lens, and the diagram looks like Figure F.1b. But now we know that the object distance is $x_o = f_1$, and the image distance is $x_i = f_2$. The lens equation for this effective lens is (Eq. E3):

$$\frac{1}{x_o} + \frac{1}{x_i} = \frac{1}{f}$$

where f is the effective focal length of the combined lens. With the known values of the object and image distances, this becomes:

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f} \qquad \text{(F1)}$$

But $(1/f_1) = P_1$ is the *power* of the first lens; $(1/f_2) = P_2$ is the power of the second lens; and $(1/f) = P$ is the power of the combined lens. So Equation F1 becomes:

$$P_1 + P_2 = P \qquad \text{(F2)}$$

—the *powers add* for the two touching lenses, as advertised.

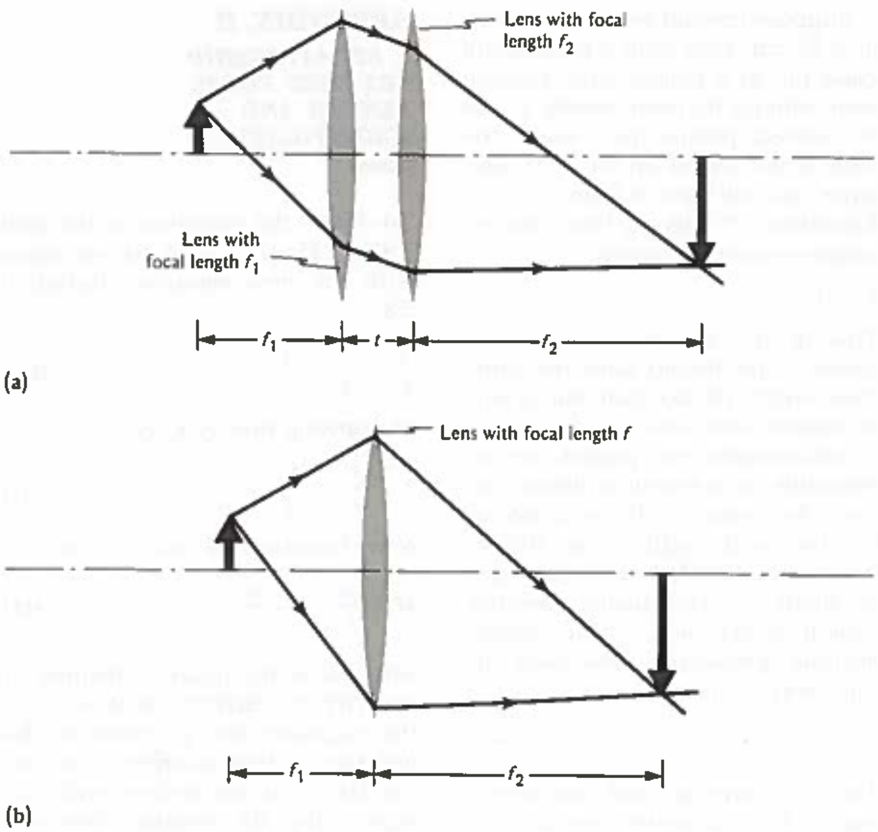


FIGURE F.1

(a) Two thin lenses separated by a distance t. (b) Two thin lenses touching can be thought of as one effective lens.

APPENDIX G
PHOTOGRAPHIC PERSPECTIVE

When you look at a scene, each object in your field of view produces an image of a particular size on your retina. For the perspective in a *photograph* of the scene to appear correct to you, the retinal image sizes must be the same as when you viewed the *scene* directly. Another way to say this is that the visual angle subtended by each image on the photograph must be the same as that subtended by the corresponding original object. Clearly this condition depends on the focal length of the lens used for photographing the scene, the magnification of any resulting print, and the distance from which you view the photograph.

Figure G.1a shows a camera photographing a distant object in a scene. Recall that the relation between the object size, s_o, and the image size, s_i, depends only upon the object distance, x_o, and the image distance, x_i. As Equation E5 states:

$$\frac{s_i}{s_o} = -\frac{x_i}{x_o} \quad \text{(E5)}$$

For distant objects, the image lies in the focal plane of the converging lens being used, hence $x_i = f_c$. Here f_c is the focal length of the camera lens. Equation E5 is then:

$$s_i = -f_c \frac{s_o}{x_o} \quad \text{(G1)}$$

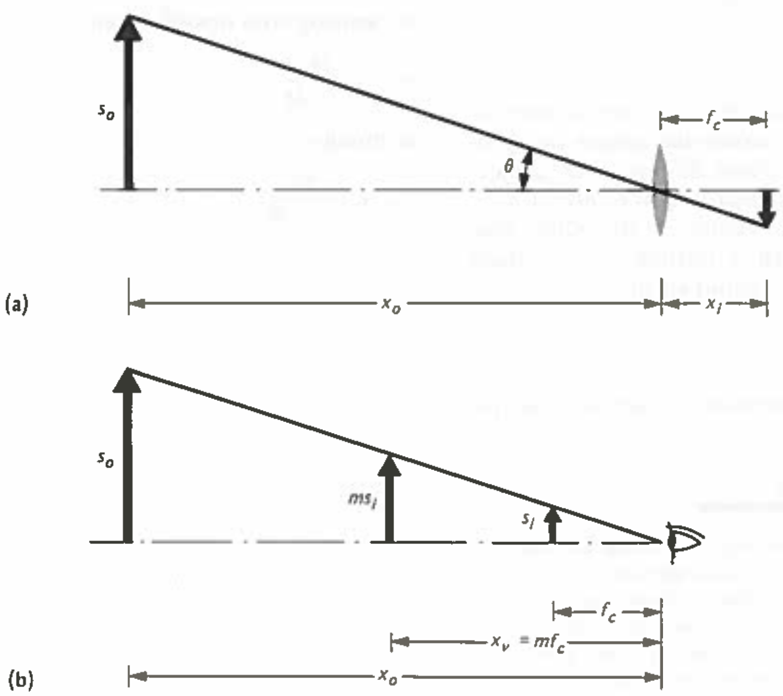


FIGURE G.1

(a) A camera whose lens has focal length f_c photographs a distant object whose size is s_o. The object subtends an angle θ and produces an (inverted) image of size $-s_i$ on the film. (b) An eye (represented by a lens) views a photographic image of the original object. If there is no magnification ($m = 1$), the eye should view the photo from a distance of f_c for the image to subtend the same angle θ as in (a). If the photo is enlarged by a factor m, then the eye must view it from a distance mf_c. This will guarantee the proper perspective.

Thus f_c sets the *scale* of the sizes of the images; all image sizes are proportional to f_c. Hence your telephoto lens gives you a larger image.

Suppose you now look directly at the developed photographic image. Figure G.1b shows that you must view it from a distance f_c in order that each photographic image subtends the same angle as the corresponding original object did; that is, so that the perspective appears normal. Usually you can't focus from so close, so you must enlarge the image.

Suppose you enlarge your photograph, *increasing its size* by a magnification factor m. As Figure G.1b shows, you must now look at the print from a *larger viewing distance* for the photographic image to subtend the same angle and hence the perspective to appear correct. As the figure illustrates, the proper viewing distance is:

$$x_v = mf_c \qquad \text{(G2)}$$

Note that this result is independent of the size and distance of the object in the scene. That is, if you view your print from the proper distance for one of the objects, *all* objects will appear the proper size and hence the perspective will appear correct. If you view the print from a smaller distance, it will have telephoto perspective (see the TRY IT for Sec. 4.3B); if you view it from a larger distance, it will have wide-angle perspective.

Let's consider some applications of Equation G2. Suppose first you take a 35-mm picture and blow it up to 12 × 18 cm (a 5 × 7" print). This means a magnification of about $m = 5$ (since a 35-mm negative is 24 × 36 mm). If you wish to view this print from 25 cm, what focal-length camera lens should you have used so the perspective looks correct? According to Equation G2, correct perspective is achieved if:

$$f_c = \frac{x_v}{m}$$

and $x_v = 25$ cm while $m = 5$, so $f_c = 5$ cm $= 50$ mm. Thus, for these conditions, a 50-mm camera lens gives correct perspective.

Suppose instead you wish to look at a 35-mm slide with a magnifying glass (as in a pocket slide viewer); what should its focal length, f_m, be for correct perspective? Since the slide is not blown up ($m = 1$), and since you will view it from $x_v = f_m$, Equation G2 tells us that you get proper perspective when:

$$f_c = f_m$$

That is, the magnifying glass and camera lens should have the same focal length. (Notice that this is true no matter what size film you use.)

Now suppose you project the 35-mm slide on a screen a distance x_p from the *projector*. If the projector lens has focal length f_p, then the object distance (slide to projector lens) is about f_p. The image distance (lens to screen) is x_p, so the magnification produced (described by Equation E5) is:

$$m = \frac{x_p}{f_p} \qquad \text{(G3)}$$

(Here we have ignored the minus sign in Eq. E5, which tells us that the image is inverted—that's why you put the slide in upside down.) If you view the projected image from a distance x_v from the screen, using the value of m in Equation G3 tells us that the perspective will be correct providing:

$$x_v = x_p \frac{f_c}{f_p}$$

This is the distance you should sit from the screen for proper perspective. A normal 35-mm slide projector has a focal length of about f_p = 100 mm. Hence if the slide was taken with a normal f_c = 50-mm lens, you should sit at:

$$x_v = \frac{x_p}{2}$$

halfway between the screen and the projector.

PONDER

Suppose the original photograph was taken with a wide-angle lens, say f_c = 25 mm. Where should you sit (if proper perspective were the only consideration)? Suppose it was taken with a 200-mm telephoto lens?

APPENDIX H
A RELATIONSHIP BETWEEN FOCAL LENGTH AND MAGNIFICATION

To derive the equation of the first TRY IT for Section 4.4B, we begin with the *lens equation*, Equation E3:

$$\frac{1}{x_o} + \frac{1}{x_i} = \frac{1}{f} \qquad \text{(E3)}$$

Multipyling this by x_o gives:

$$1 + \frac{x_o}{x_i} = \frac{x_o}{f} \qquad \text{(H1)}$$

Now, Equation E5 tells us that:

$$M = \frac{x_i}{x_o} = -\frac{s_i}{s_o} \qquad \text{(H2)}$$

where M is the quantity defined in the TRY IT. (Strictly, if M is to be the *magnification*, it should be the negative of this quantity: s_i/s_o. In the TRY IT it was defined with the sign of Eq. H2 because there you would measure all distances as *positive*, despite the fact that the image is inverted.) Using Equation H2 in Equation H1, we get:

$$\frac{x_o}{f_o} = 1 + \frac{1}{M} = \frac{M+1}{M}$$

or, writing this upside down:

$$\frac{f}{x_o} = \frac{M}{1+M}$$

or, finally:

$$f = x_o \frac{M}{1+M}$$

APPENDIX I
LOGARITHMS

Ordinary scales used for graphs, such as that shown in Figure I.1a, have equal values between markings. In the example shown, when you go from one marking to the next, you *add* 1, no matter which marking you started at. Such a scale is called a *linear* scale. However, we have seen cases where such a scale would be very inconvenient. For example, in Table 1.1, if we had let a distance on the paper of 1 mm represent 100 Hz, we would have needed a piece of paper that stretched from here to beyond the star Sirius in order to show the frequency range included in the table.

We managed to include such a tremendous range by using a scale in which each step differed from the previous one by a *multiplicative factor* of 10 (Fig. I.1b). Here the second marking corresponds to a number 10 times as big as the first, the third to a number 10 times as big as the second, and so on. We've indicated the marking numbers in the powers-of-ten notation, and you see that the markings correspond to equal steps in the *power*—from 10^1 to 10^2 to 10^3, etc.

The *logarithm* is a device to display that power. The logarithm of a number y, written as $\log y$, is defined as follows:

$\log y = x$ means $10^x = y$

Thus, the logarithm of the markings in Figure I.1b would be $\log 10 = 1$, $\log 100 = 2$, . . . , because $10^1 = 10$, $10^2 = 100$, ... The markings, then, correspond to equal steps on a *logarithmic* (or *log*) scale.

Logarithmic scales find use in many places where you are interested in the percentage change of something rather than the actual value. A common example, from the newspaper, is the stock market (Fig. I.2). It doesn't matter whether your stock cost $1 per share or $100 per share; if you bought $5000 worth of shares and their value doubled, your shares would then be worth $10,000 in either case. On a linear scale, however, an increase from $1 to $2 per share would look very small compared to an increase from $100 to $200 per share. On a log scale, the size of the increase would be the same in both cases because each increase was a *factor* of 2. The log scale enables you to compare how well different stocks are doing with respect to each other, even though their prices may be quite different.

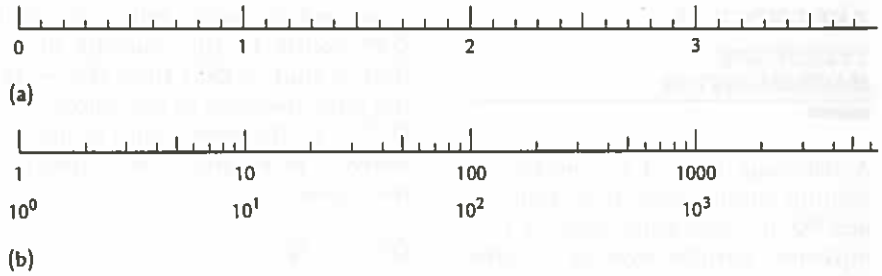


FIGURE I.1

(a) In a linear scale, successive equal increments correspond to the addition of a constant. **(b)** In a logarithmic scale, successive equal increments correspond to the multiplication by a constant.

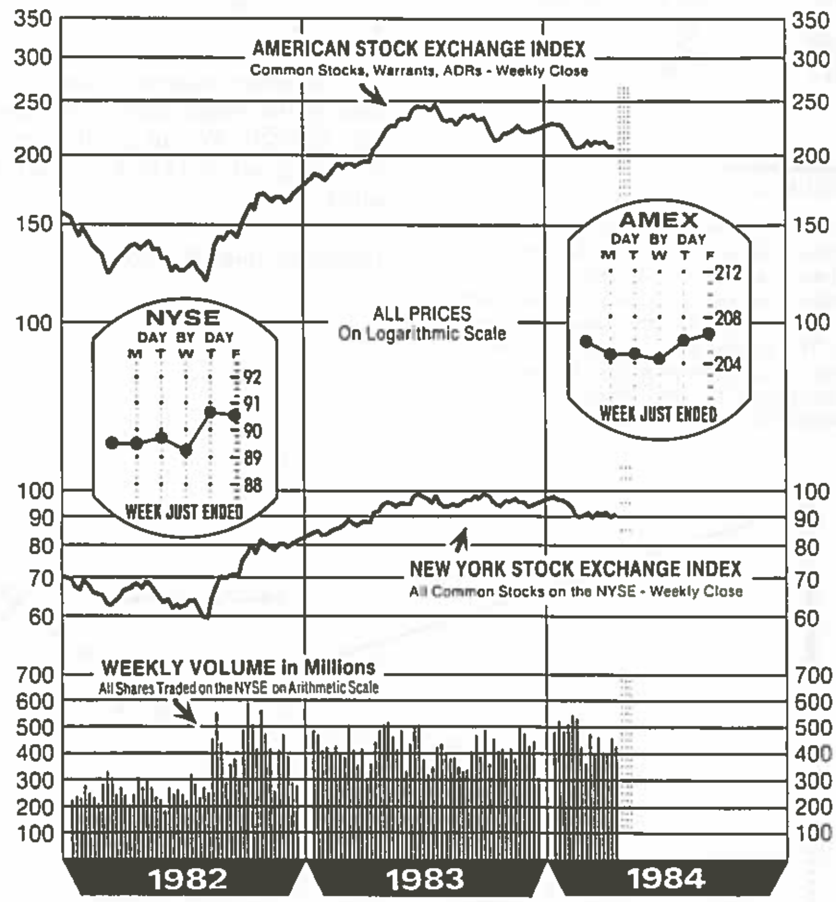


FIGURE I.2

Stock Exchange indexes plotted on a logarithmic scale versus time, which is plotted on a linear scale.

APPENDIX J

TELESCOPE MAGNIFICATION

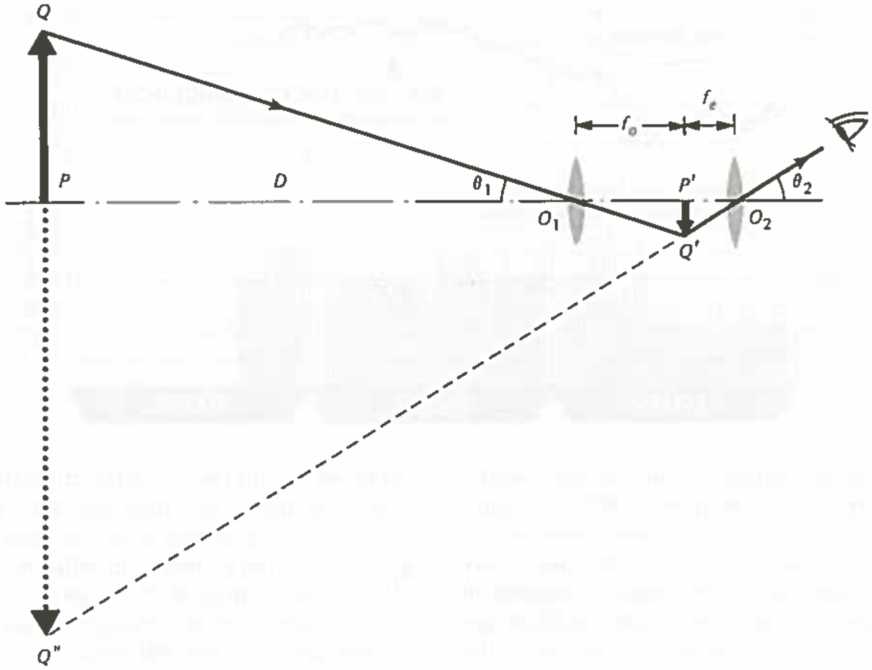
A telescope (Fig. J.1) converts incoming parallel rays, from some object PQ at a very large distance D, to outgoing parallel rays at a different, larger, angle. An eye looking through the telescope then sees the distant virtual image PQ'' still at a large distance, but subtending a larger angle than the object did without the telescope (that is, $\theta_2 > \theta_1$).

In the figure, we see pairs of similar triangles: ΔQPO_1 and $\Delta Q'P'O_1$, as well as $\Delta Q''PO_2$ and $\Delta Q'P'O_2$. The first pair yields the relation:

$$\frac{\overline{QP}}{\overline{Q'P'}} = -\frac{\overline{PO_1}}{\overline{O_1P'}}$$

FIGURE J.1

A simple telescope consisting of two converging lenses is used to view a distant object PQ. The image $P'Q'$ is behind the first lens (the objective lens), a distance equal to the lens' focal length, f_o. The second lens (the eyepiece) is used as a magnifying glass for viewing the image $P'Q'$, and forms a virtual image PQ'' very far away.



(The minus sign indicates that $\overline{Q'P'}$ points in the opposite direction to that of $\overline{QP}$.) Here $\overline{PO_1} = D$, the large distance to the object; and $\overline{O_1P'} = f_o$, the focal length of the objective. Rewriting this equation then gives:

$$\overline{QP} = -\overline{Q'P'}\,\frac{D}{f_o} \qquad \text{(J1)}$$

The second pair of triangles yields:

$$\frac{\overline{Q''P}}{\overline{Q'P'}} = \frac{\overline{PO_2}}{\overline{P'O_2}}$$

where $\overline{P'O_2} = f_e$, the focal length of the eye piece; and $\overline{PO_2} = D + f_o + f_e \simeq D$, if D is large compared to the focal lengths f_o and f_e. Hence, this equation gives:

$$\overline{Q''P} = \overline{Q'P'}\,\frac{D}{f_e} \qquad \text{(J2)}$$

The telescope magnification is the ratio of the image size to the object size: $\overline{Q''P}/\overline{QP}$. We can get this by dividing Equation J2 by Equation J1, which gives:

$$\text{Telescope magnification} = -\frac{f_o}{f_e} \qquad \text{(J3)}$$

APPENDIX K

POSITIONS OF INTERFERENCE AND DIFFRACTION FRINGES

Let's derive the *locations* of the intensity maxima (bright fringes) and minima (dark fringes) in the screen patterns considered in Chapter 12. Maxima occur at points of constructive interference (waves in phase) and minima at points of destructive interference (waves out of phase). We shall assume that all the *sources* are *in phase.* A phase difference then occurs only when the interfering beams travel *different distances* to reach a point on the screen. Recall that the *waves* from two sources are:

in phase
 if their path difference ϵ is
 $0, \pm\lambda, \pm 2\lambda, \pm 3\lambda, \ldots$ (K1)

out of phase
 if their path difference ϵ is
 $\pm\lambda/2, \pm 3\lambda/2, \pm 5\lambda/2, \ldots$

PONDER

If the two sources are out of phase, however, the out-of-phase case and the in-phase case are interchanged in Equation K1. That is, maxima become minima and vice versa. Why?

1. Two Sources. The path difference ϵ can be computed by plane geometry. For example, let's examine the case of *two sources* as shown in Figure K.1. In this figure we can consider a point P a distance h to the right of the center point O on the screen. (A negative h means that P is to the left of O.) We can now find ϵ by similar triangles. The two angles marked ϕ are equal, because they are alternate interior angles for parallel lines (the screen and the line connecting the sources). The angles marked θ are therefore also equal, because they are complementary to the angles ϕ. From Appendix B, where the sine function was defined, we have (using ΔS_1S_2Q):

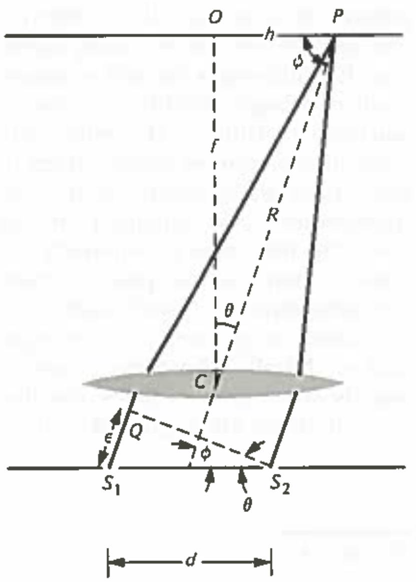


FIGURE K.1

Geometry of the rays from two sources S_1 and S_2 that interfere in the focal plane of a lens of focal length f, centered at C.

$$\frac{\epsilon}{d} = \sin\theta$$

or:

$$\epsilon = d\sin\theta \qquad \text{(K2)}$$

We can find $\sin\theta$ by using ΔCOP:

$$\sin\theta = \frac{h}{R}$$

In many interference set-ups the angle θ is quite small, so that $R \approx f$, and hence $\sin\theta \approx h/f$. Equation K2 then gives:

$$\epsilon = \frac{dh}{f} \text{ (for small } \theta) \qquad \text{(K3)}$$

Now we can apply the criterion of Equation K1. There are maxima when:

$$\frac{dh}{f} = 0, \pm\lambda, \pm 2\lambda, \pm 3\lambda, \ldots$$

that is, for points on the screen where:

$$h = 0, \pm\lambda\frac{f}{d}, \pm 2\lambda\frac{f}{d}, \pm 3\lambda\frac{f}{d}, \ldots \qquad \text{(K4)}$$

(position of intensity maxima)

Similarly, we find the minima when:

$$h = \pm\lambda\frac{f}{2d}, \pm 3\lambda\frac{f}{2d}, \pm 5\lambda\frac{f}{2d}, \ldots \qquad \text{(K5)}$$

(position of intensity minima)

The fringe spacing is the distance between successive maxima. Hence:

$$\textbf{fringe spacing} = \lambda\frac{f}{d} \qquad \text{(K6)}$$

If the focal length f of the lens is very large, the lens is very weak, and we can simply remove it without significant change in the geometry, leaving the screen at the same *large* distance, which we now call D. Equation K6 then says that the fringes spacing is $\lambda D/d$—the formula used in Section 12.2E.

Note, however, that for high order fringes, the angle θ is *not* small, and we *cannot* replace R by f. In fact, since $\sin\theta$ never exceeds 1 (no matter what the angle θ), Equation K2 shows that ϵ can never exceed d. Hence there can only be a finite number of fringes on the screen, and our approximate formula, Equation K6, for the fringe spacing ceases to be valid for high orders. (The fringe spacing increases for higher order fringes, and the total number of fringes is finite.)

2. Thin Film. Equation K1 for two-beam interference can also be used to find the wavelengths that give intensity maxima and minima when light is *reflected by a thin film*. We assume that the first and second reflections are both hard or both soft. (If one is hard and the other soft, we need only interchange "maxima" and "minima" in the following.) If the thickness of the film is t, the extra path length ϵ of the beam reflected from the second surface is $2t$ (at near normal incidence). Hence we get:

maxima at

$$\lambda' = 2t, \frac{2t}{2}, \frac{2t}{3}, \frac{2t}{4}, \frac{2t}{5}, \ldots \qquad \text{(K7)}$$

minima at

$$\lambda' = 4t, \frac{4t}{3}, \frac{4t}{5}, \frac{4t}{7}, \ldots$$

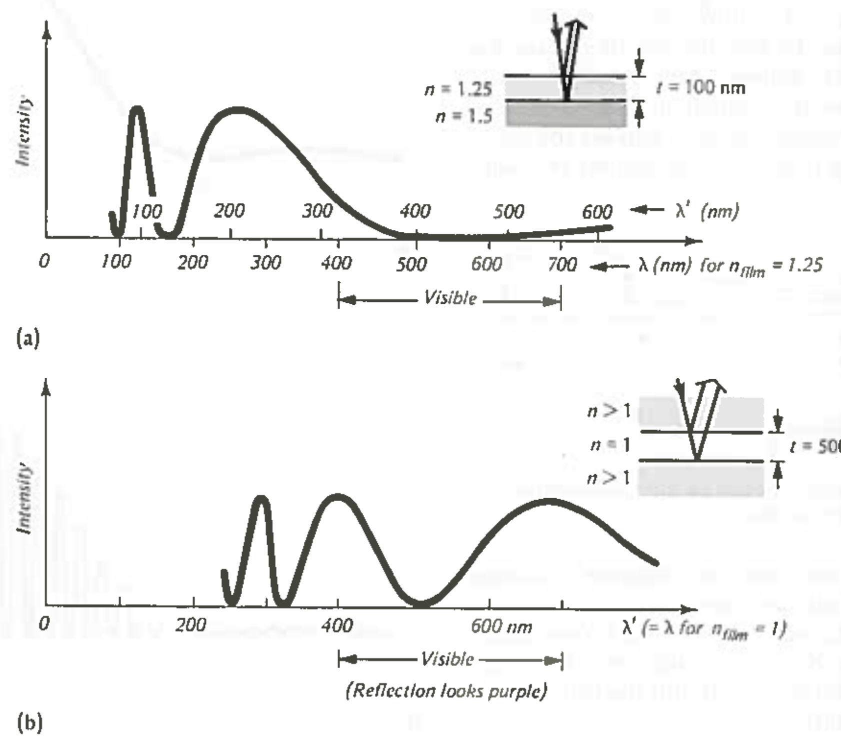


FIGURE K.2

Reflected intensity versus wavelength for **(a)** a film with both reflections in phase, **(b)** a different film with reflections out of phase. The inserts in the graph show the arrangement of the film.

Here λ' is the wavelength *in* the film. To relate λ' to λ, the wavelength in air, we recall that when a light wave enters a medium of index of refraction n, the frequency of the wave is not changed (Sec. 1.3A), but the speed of the wave changes from c to c/n (Sec. 2.5). From the formula of Section 1.3A that relates speed and frequency to wavelength we then find $\lambda' = \lambda/n$.

Thus, for a given thickness, Equation K7 tells us those wavelengths that are most strongly reflected (maxima), and those that are not reflected (minima). At intermediate wavelengths the reflected intensity varies smoothly between these maxima and minima. For example, Figure K.2a shows this distribution for t = 100 nm, a typical thickness for an antireflective coating where the first and second reflections are both hard. Shown in Figure K.2b is the distribution for t = 500 nm in the case where the first reflection is soft and the second hard.

3. Wedge. If the light is *monochromatic* we can use Equation K7 to find the film *thicknesses* t that give maxima or minima. In a film of variable thickness, then, Equation K7 tells us the positions of the maxima and minima. For example, consider the film formed by the *wedge* of air between two plane sheets of glass that touch on one edge and are separated by a gap on the other edge (Fig. K.3). By similar triangles

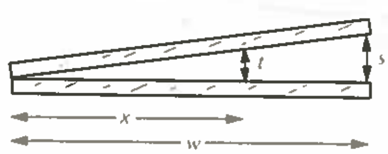


FIGURE K.3

A wedge-shaped air film between two sheets of glass.

we see that the distance x across the air film satisfies $x/t = w/s$. Substituting for t the values from Equation K7 and noting that here one reflection is soft and the other hard, we find: minima at

$$x = 0, \frac{\lambda w}{2s}, 2\frac{\lambda w}{2s}, 3\frac{\lambda w}{2s}, \ldots \quad \text{(K8)}$$

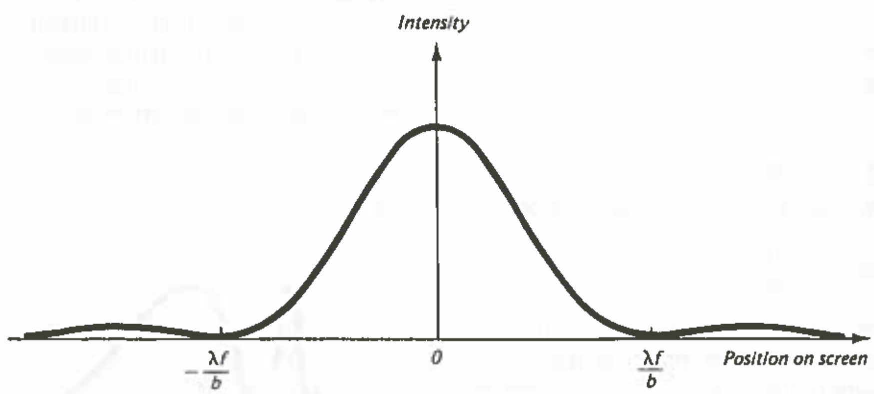
(Since the wedge is made of air, we have here $n = 1$ and $\lambda' = \lambda$.) The minima appear as dark fringes (parallel to the edge of the wedge), and there are bright fringes in between.

How many fringes are there across the entire width, that is, when $x = w$? Since x increases by $(\lambda w/2s)$ for each fringe, there will be a total number of $w/(\lambda w/2s) = 2s/\lambda$ fringes. For example, if s = 0.1 mm and λ = 500 nm, we find that there are 400 fringes. Changing s by only $\lambda/2$ = 250 nm changes the number of fringes by one, a change that is easily observable. Thus such a small distance can be very accurately measured by this easily visible effect.

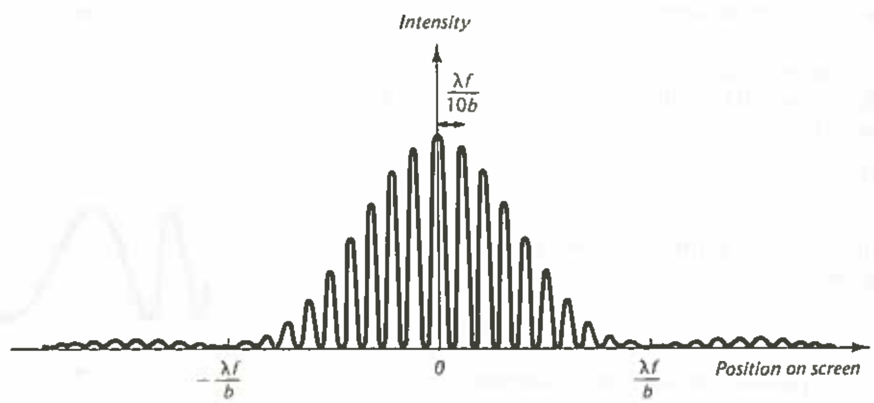
4. Single slit. The intensity minima in the Fraunhofer diffraction pattern of a *single slit* of width b can also be treated by using Equation K5. Although the slit is equivalent to a large (infinite) number of sources according to Huygens' principle, these sources must cancel in *pairs* in order to obtain destructive interference. As explained in the text, the first zero of intensity occurs at such a screen position that, for each source in the left half of the slit, there is a source in the right half of the slit whose wave arrives exactly out of phase. Since the distance between such pairs of sources

FIGURE K.4

(a) Intensity versus screen position in the Fraunhofer diffraction pattern of a slit of width b. **(b)** Intensity versus screen position for Young's fringes when the separation of the two slits is ten times the width of the individual slits. (The same slit width is used in both plots.)



(a)



(b)

is $d = b/2$, this value substituted in Equation K5 gives the first minimum (zero) of intensity for the slit:

$$h = \pm\lambda\frac{f}{b} \tag{K9}$$

(Again, for a distant screen we can remove the lens and replace f by D. This gives the formula of Sec. 12.5A.) We get further minima whenever we can divide the slit width b into an *even* number of regions such that each Huygens' source in one region has a destructively interfering partner in the next region. This corresponds to pairs of sources at distances $d = b/2$ (as in Eq. K9), or $d = b/4$, or $d = b/6$, . . . which, by Equation K5, have their *first minima* (in the small-angle approximation) at:

$$\begin{aligned} h &= \pm\frac{\lambda f}{\frac{2b}{2}}, \pm\frac{\lambda f}{\frac{2b}{4}}, \pm\frac{\lambda f}{\frac{2b}{6}}, \ldots \\ &= \pm\frac{\lambda f}{b}, \pm\frac{2\lambda f}{b}, \pm\frac{3\lambda f}{b}, \ldots \end{aligned} \tag{K10}$$

This equation gives the locations of *all* the minima in the Fraunhofer diffraction pattern of a slit of width b. (The higher-order minima for any one of the source distances are already contained in this sequence, so they do not have to be listed separately.)

Thus the dark fringes due to a single slit are regularly spaced about the origin O by $\lambda f/b$, except at the origin itself ($h = 0$). The central fringe is of course bright, because all the sources contribute in phase there. Other bright fringes occur between the dark fringes. Unlike in the Young's fringe pattern, these single-slit side maxima have intensities that *decrease* rapidly with h. For example, the intensity of the first bright fringe near the central fringe (at $h = 3\lambda f/2b$) is less than $\frac{1}{20}$ of that of the central fringe. Figure K.4a shows a plot of the intensity of the single-slit Fraunhofer diffraction pattern. Figure K.4b shows how this diffraction pattern modifies the Young's fringes produced by two slits.

APPENDIX L

BREWSTER'S ANGLE

To find Brewster's angle, we must find the incident angle for the special case when the transmitted beam is perpendicular to the reflected beam. Figure 13.5 tells us that this happens when the reflection angle, θ_r, and the transmission angle, θ_t, satisfy the relation:

$$\theta_r + 90° + \theta_t = 180°, \quad \text{or}$$

$$\theta_t = 90° - \theta_r$$

Since the reflection angle equals the incidence angle ($\theta_r = \theta_i$), this means that we are at Brewster's angle when:

$$\theta_t = 90° - \theta_i \tag{L1}$$

But Snell's law (Eq. B5) gives another relation between θ_t and θ_i:

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{B5}$$

Therefore, at Brewster's angle both Equations L1 and B5 must be true. Using Equation L1, we can then rewrite Equation B5 as:

$$\frac{\sin\theta_i}{\sin(90° - \theta_i)} = \frac{n_t}{n_i} \tag{L2}$$

Recall how the sine of an angle is defined (Eq. B1) and consider Figure L.1. From this figure you can see that:

$$\sin\theta_i = \frac{p}{r}$$

and

$$\sin(90° - \theta_i) = \frac{q}{r}$$

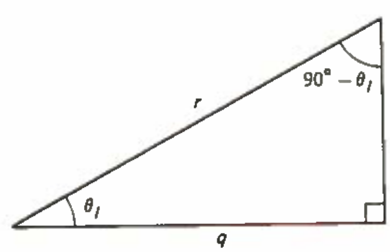


FIGURE L.1

A right triangle with one angle equal to θ_i.

Hence:

$$\frac{\sin\theta_i}{\sin(90° - \theta_i)} = \frac{p}{q} \tag{L3}$$

The ratio p/q represents another trigonometric function, the ***tangent*** of θ_i:

$$\tan\theta_i = \frac{p}{q} \tag{L4}$$

Hence, when the angle of incidence is Brewster's angle ($\theta_i = \theta_B$), we see from Equations L4, L3, and L2, that:

$$\tan\theta_B = \frac{p}{q} = \frac{\sin\theta_i}{\sin(90° - \theta_i)} = \frac{n_t}{n_i}$$

or:

$$\mathbf{\tan\theta_B = \frac{n_t}{n_i}} \tag{L5}$$

As an example, for light traveling *from air* ($n_i = 1.0$) *to water* ($n_t = 1.3$), Equation L5 and a pocket calculator tells us that $\theta_B = 52.4°$. For light traveling *from air to glass* ($n_t = 1.5$), Brewster's angle is $\theta_B = 56.3°$.

APPENDIX M
MALUS'S LAW

We want to know how the intensity of the transmitted light varies as the angle between the polarizer and the analyzer is changed. The polarizer transmits only light polarized in one direction (arbitrarily drawn vertical in Fig. M.1a). This polarized light is then incident on the analyzer. Suppose the analyzer is oriented at an angle θ with respect to the polarizer. We must then break up the electric field incident at the analyzer (E_{in}) into two components: one that the analyzer will pass (E_{out}), and a perpendicular component that it won't.

From Figure M.1a we see that:

$$\frac{E_{in}}{E_{out}} = \cos\theta \qquad \text{(M1)}$$

where the ***cosine*** of an angle is defined, using Figure L.1, as:

$$\cos\theta_i = \frac{q}{r} \qquad \text{(M2)}$$

Since the intensity of light is proportional to the *square* of the electric field, the intensity of the light transmitted by the analyzer is seen to be (upon squaring Eq. M1):

$$\frac{I_{in}}{I_{out}} = \cos^2\theta \qquad \text{(M3)}$$

Equation M3 is called ***Malus's law***, and is plotted in Figure M.1b.

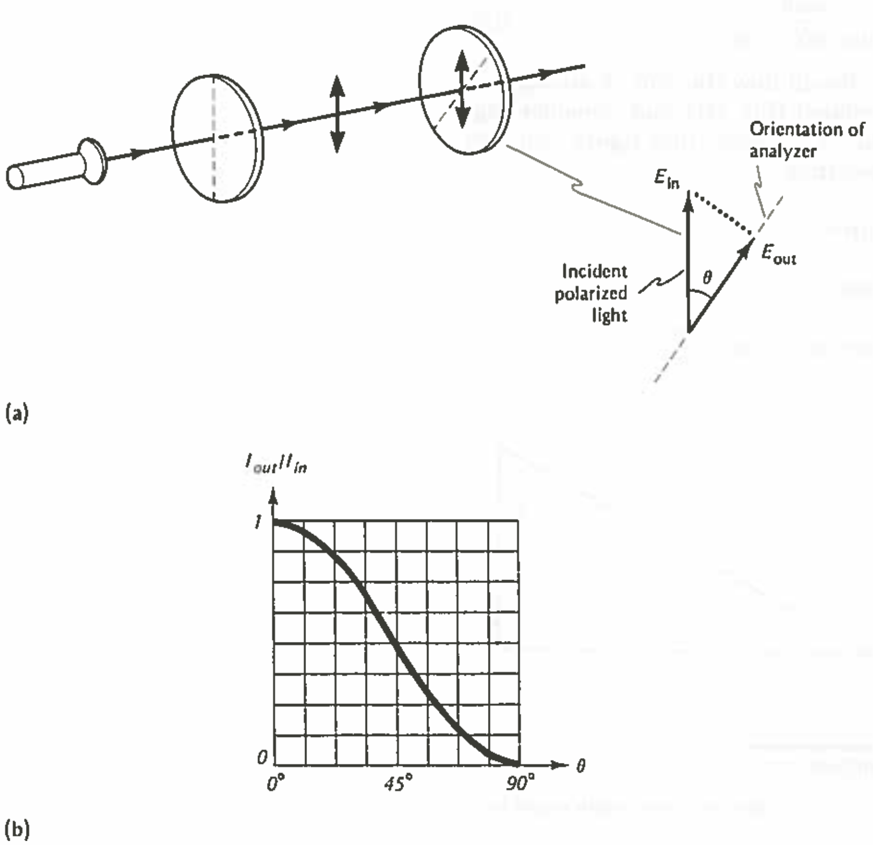


FIGURE M.1

(a) Only the component of the incident polarized light that is parallel to the analyzer's orientation will be transmitted by the analyzer. (b) A graph showing the ratio of the intensity of the light transmitted by the analyzer to that of the light incident on the analyzer, for different values of the angle θ between the direction of polarization of the incident light and the orientation of the analyzer.

APPENDIX N
HOLOGRAPHIC RECORDING AND RECONSTRUCTION OF WAVES

Let's check that a transmission hologram such as described in Figure 14.3 indeed forms a plane reconstructed beam at the *same* angle as that of the original object beam. Figure N.1a shows plane reference and object beams striking the film. The reference beam strikes the film perpendicularly and is shown at an instant when one of its crests lies along the thin emulsion. The object beam arrives at angle θ_o, so $\angle OAB = \theta_o$. The wavelength of the light in both of these beams is λ. The regions of constructive interference on the film (the bright fringes) are spaced a distance *d*. Using Δ*OAB* and the definition of the sine (Eq. B1), we can see that *d*, λ, and θ_o are related in the following way:

$$\sin\theta_o = \frac{\lambda}{d} \qquad \text{(N1)}$$

During reconstruction (Fig. N.1b), the hologram acts as a diffraction grating. The wavelength of the reconstruction beam is λ, the same as that used for exposure. The grating constant here is *d*, the same value as the fringe spacing during exposure. We get constructive interference when the path difference between two different slits is equal to a wavelength. That is, according to Equations K1 and K2, we get a first-order beam when:

$$\epsilon = \pm\lambda = d\sin\theta_{rec} \qquad \text{(N2)}$$

or:

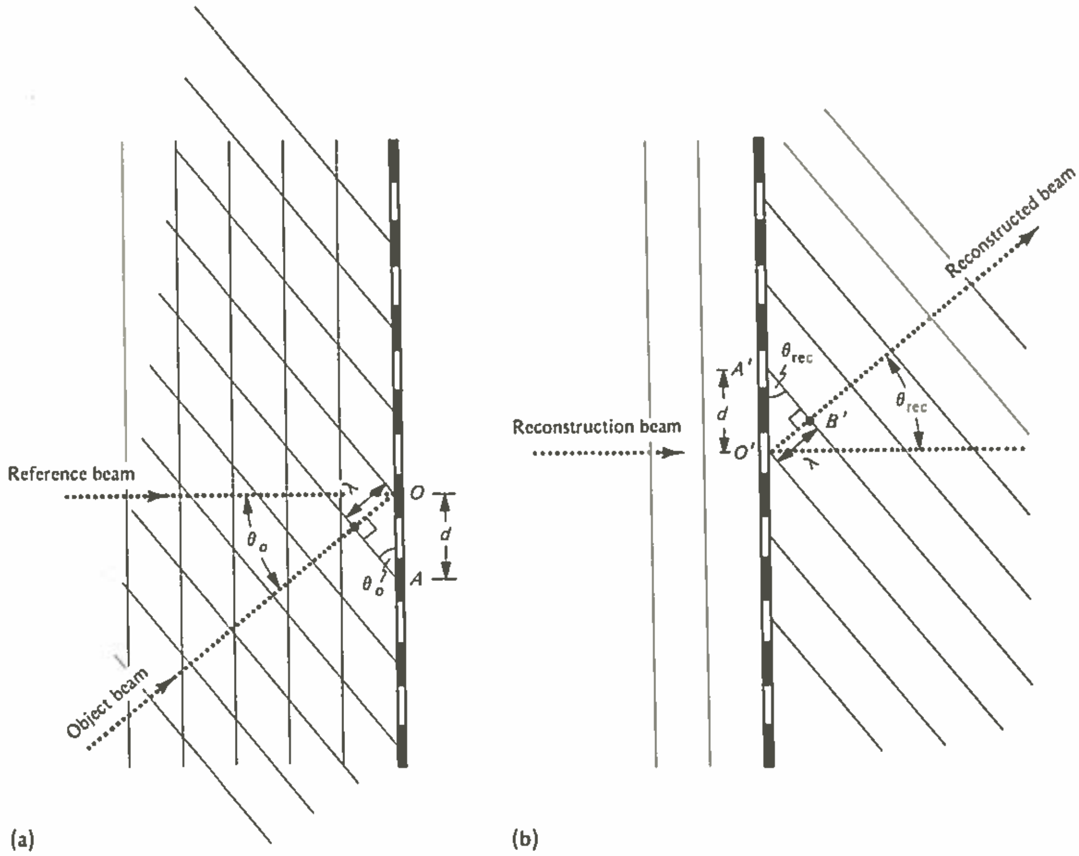
$$\sin\theta_{rec} = \frac{\pm\lambda}{d} \qquad \text{(N3)}$$

Comparing Equations N1 and N3 gives, for the + sign in Equation N3:

$$\sin\theta_{rec} = \sin\theta_o \qquad \text{(N4)}$$

or:

$$\theta_{rec} = \theta_o$$



(a) (b)

FIGURE N.1

(a) Exposure of film by two plane waves. The reference beam arrives perpendicularly to the film while the object beam arrives at angle θ_o.
(b) During reconstruction, the first order (reconstructed) beam leaves the hologram at angle θ_{rec}, where $\theta_{rec} = \theta_o$.

Hence the reconstructed beam leaves the hologram at the same angle as that of the object beam.

PONDER

The − sign in Equation N3 gives $\theta_{rec} = -\theta_o$. What does this mean?