EXAMINATION
MATHEMATICS 391
Due: L306, High Noon, Wednesday, December 16, 2015
NO LIVING SOURCES

01• Let $(\Omega, \mathcal{A}, \pi)$ be a probability space and let F be a real valued random variable defined on Ω. Let the distribution of F be the standard normal ρ:

$$\rho = F_*(\pi)$$

By definition:

$$\rho((u,v)) = \frac{1}{\sqrt{2\pi}} \int_u^v exp(-\frac{1}{2}y^2)dy$$

where u and v are any numbers for which $u < v$. Let G be the nonnegative real valued random variable defined on Ω by squaring F:

$$G = F^2$$

Find the distribution σ of G:

$$\sigma = G_*(\pi)$$

02• Let Ω be the closed unit disk in $\mathbf{R}^2$, having center $(0,0)$ and radius 1:

$$(x,y) \in \Omega \text{ iff } x^2 + y^2 \leq 1$$

Let Ω be supplied with the probability measure π, defined by normalization of the measure of area:

$$\pi(A) = \frac{1}{\pi} \int \int_A 1 \cdot dxdy$$

where A is any reasonable subset of Ω. Let α and β be the associated projection mappings carrying Ω to the closed finite interval $[-1,1]$ in $\mathbf{R}$:

$$\alpha(x,y) = x, \quad \beta(x,y) = y$$

where (x,y) is any member of Ω. Of course, one may interpret α and β as random variables. Describe the joint distribution and the marginal distributions for α and β:

$$(\alpha \times \beta)_*(\pi), \quad \alpha_*(\pi), \quad \beta_*(\pi)$$

Are α and β independent?

03• Consider the following transition matrix Π for a Markov Process:

$$\Pi = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Find a probability measure:

$$P = (P_1, P_2, P_3, P_4)$$

meeting the Condition of Invariance:

$$P\,\Pi = P$$

Determine whether or not the corresponding Markov Process is ergodic.

04• Let $(\Omega, \mathcal{A}, \pi)$ be a probability space and let $\mathcal{F}$ be a real valued random process defined on Ω:

$$\mathcal{F}: \quad F_0, F_1, F_2, \ \dots\ , F_j, \ \dots$$

Let the process be independent and identically distributed. Let the common mean and variance be 0 and 1, respectively. (We assume, implicitly, that the common variance is finite and that the Standard Maneuver has been applied.) For each positive integer n, let $\bar{F}_n$ be the average of the first n terms of $\mathcal{F}$:

$$\bar{F}_n = \frac{1}{n}(F_0 + F_1 + \ \cdots\ + F_{n-1})$$

Of course, the mean and variance of $\bar{F}_n$ are:

$$0, \ \frac{1}{n}$$

respectively. Now let j be any positive integer. Let $A_{j,n}$ be the subset of Ω consisting of all members ξ for which:

$$(*) \qquad |\bar{F}_n(\xi)|^2 \leq (\frac{1}{j})^2$$

Apply Chebychev's Inequality to show that:

$$1 - \frac{j^2}{n} \leq \pi(A_{j,n})$$

05$^\bullet$ Let X be the set consisting of the eight members:

$$(j, k, \ell) \qquad (j, k, \ell \in \{0, 1\})$$

The members of X are the vertices of the unit cube in $\mathbf{R}^3$. Let $\mathcal{A}$ be the borel algebra consisting of all subsets of X. Let μ be the measure on $\mathcal{A}$ defined by the following relations:

$$\mu(\{(0,0,0)\}) = \mu(\{(1,1,0)\}) = \mu(\{(1,0,1)\}) = \mu(\{(0,1,1)\}) = \frac{1}{4}$$

$$\mu(\{(1,0,0)\}) = \mu(\{(0,1,0)\}) = \mu(\{(0,0,1)\}) = \mu(\{(1,1,1)\}) = 0$$

Let f, g, and h be the random variables defined on X as follows:

$$f((j,k,\ell)) = j, \;\; g((j,k,\ell)) = k, \;\; h((j,k,\ell)) = \ell \qquad ((j,k,\ell) \in X)$$

Describe:

$$f_*(\mu), \; g_*(\mu), \; h_*(\mu)$$

and:

$$(f \times g)_*(\mu), \; (f \times h)_*(\mu), \; (g \times h)_*(\mu)$$

Verify that f and g are independent, that f and h are independent, that g and h are independent, but that f, g, and h are not independent.

06$^\bullet$ Let n be a positive integer. Let P be a probability measure on the set:

$$\{1, 2, 3, \;\ldots\;, n\}$$

We display P as follows:

$$P = (P_1, P_2, P_3, \;\ldots\;, P_n)$$

where:

$$(*) \qquad 0 \leq P_j \;\; (1 \leq j \leq n) \;\text{ and }\; \sum_{j=1}^{n} P_j = 1$$

One defines the Entropy of P by the following expression:

$$\eta(P) = -\sum_{j=1}^{n} P_j log(P_j)$$

Find the maximum value of η, subject to conditions $(*)$.