MATHEMATICS 391
ASSIGNMENT 11
Due: December 02, 2015

01° Let n be a positive integer. Let Π be a (nonempty) compact convex subset of $\mathbf{R}^n$. Let f be the real valued function defined on $\Pi \times \Pi$ as follows:

$$f(X, Y) = \|X - Y\| \qquad (X \in \Pi,\ Y \in \Pi)$$

Since $\Pi \times \Pi$ is a compact subset of $\mathbf{R}^n \times \mathbf{R}^n$, there must be members $\bar{X}$ and $\bar{Y}$ in Π such that $f(\bar{X}, \bar{Y})$ is the maximum value of f. Show that $\bar{X}$ and $\bar{Y}$ must be *extreme points* in Π, that is, *vertices*.

02$^\bullet$ Let $n = 9$. Let Δ be the standard simplex in $\mathbf{R}^n$ and let T be the following stochastic matrix:

$$T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

having 9 rows and 9 columns. For the corresponding Markov Chain, describe the limit set L in detail. In particular, note that L is a simplex and display its vertices. Find a member:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix}$$

of Δ such that:

$$TX = X$$

03• Let n be a positive integer and let Δ be the standard simplex in $\mathbf{R}^n$. Let H be any member of $\mathbf{R}^n$. Let ϵ and η be the real valued functions defined on Δ as follows:

$$\begin{aligned} \epsilon(X) &= \sum_{j=1}^{n} h_j x_j \\ \eta(X) &= -\sum_{j=1}^{n} x_j \, log(x_j) \end{aligned} \qquad (X \in \Delta)$$

For each member X of Δ, one may refer to $\epsilon(X)$ as the *average value* of H and to $\eta(X)$ as the *entropy*, relative to X. In turn, let $\hat{\epsilon}$ be a particular value of ϵ. Solve the following Extreme Value Problem with Constraints:

$$sup\, \eta(X) = \,? \qquad (X \in \Delta,\ \epsilon(X) = \hat{\epsilon})$$

For a proper argument you should apply some one of the methods of multi-variable calculus, for instance, the method of Lagrange Multipliers.