MATHEMATICS 391
ASSIGNMENT 09
Due: November 11, 2015

01° Let X be the set of all irrational numbers x such that $0 < x < 1$. Let T be the mapping carrying X to itself defined as follows:

$$T(x) := \frac{1}{x} - \left[\frac{1}{x}\right] \qquad (x \in X) \tag{1}$$

The ordered pair (X, T) may be viewed as a (discrete) dynamical system. One may say that, for any x in X, if the system is in state x at time 0 then the system is in state $T(x)$ one unit of time later. Note that, for any x and y in X:

$$T(x) = y \quad \text{iff} \quad (\exists j \in \mathbf{Z}^+)(x = \frac{1}{j+y})$$

Consider the (density) function w defined (by K. Gauss) on X as follows:

$$w(x) := \frac{1}{log\, 2}\,\frac{1}{1+x} \qquad (x \in X) \tag{2}$$

Let μ be the probability measure defined on X as follows:

$$\mu(E) = \int_E w(x)\lambda(dx)$$

where E is any borel subset of X. As usual, λ stands for lebesgue measure. Prove that μ is *preserved* by T, in the sense that:

$$T_*(\mu) = \mu$$

That is, prove that, for any u and v in X, if $u < v$ then:

$$T^{-1}((u,v)) = \bigcup_{j=1}^{\infty} \left(\frac{1}{j+v}, \frac{1}{j+u}\right) \tag{3}$$

and:

$$\int_u^v w(x)\lambda(dx) = \sum_{j=1}^{\infty} \int_{1/(j+v)}^{1/(j+u)} w(x)\lambda(dx) \tag{4}$$

It turns out that, supplied with the measure μ, the dynamical system (X, T) is ergodic. Consider the following observable for the system:

$$h(x) := \left[\frac{1}{x}\right] \qquad (x \in X) \tag{5}$$

Note that the values of h are positive integers. For any given x in X, consider the following discrete time series:

$$a_j(x) := h(T^j(x)) \qquad (x \in X, \quad 0 \leq j) \tag{6}$$

One refers to this series as the *continued fraction expansion (cfe)* for x. Now apply the Ergodic Theorem to show that there is a positive number K (called Khinchin's Constant) such that, for "almost every" x in X:

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} log\big(h(T^j(x))\big) = log(K)$$

That is:

$$\lim_{k\to\infty} \big(\prod_{j=0}^{k-1} a_j(x)\big)^{1/k} = K$$

Note:

$$\begin{aligned} log(K) &= \int_X log(h(x))\mu(dx) \\ &= \frac{1}{log(2)} \int_X log(\Big[\frac{1}{x}\Big])\frac{1}{1+x}\lambda(dx) \\ &= \frac{1}{log(2)} \sum_{k=1}^{\infty} log(k)\, log(\frac{(k+1)^2}{k(k+2)}) \\ &= log(2.68545\,...) \end{aligned} \tag{7}$$

so:

$$K = 2.68545...$$