MATHEMATICS 391
ASSIGNMENT 4
Due: September 30, 2015

01$^\bullet$ Let:

$$I = [0, 1]$$

be the unit interval in $\mathbf{R}$ and let λ be lebesgue measure defined on the borel subsets of I. Of course, λ is a probability measure. Let F be the random variable defined on I with values in $\mathbf{R}$, as follows:

$$F(x) = cos(2\pi x)$$

where x is any number in I. Let μ be the probability measure on the borel subsets of $\mathbf{R}$, defined as follows:

$$\mu = F_*(\lambda)$$

That is, for each borel subset E of $\mathbf{R}$:

$$\mu(E) = \lambda(F^{-1}(E))$$

By definition, μ is the distribution of F. Note that:

$$\mu(\mathbf{R} \backslash J) = 0$$

where J is the interval $J = [-1, 1]$ in $\mathbf{R}$. Why? Show that there is a borel function f defined on $\mathbf{R}$ with values in $\mathbf{R}$, such that:

$$\mu(E) = \int_E f(x)\lambda(dx)$$

where E is any borel subset of $\mathbf{R}$. Draw the graph of f. One refers to f as the *density* for μ.

02$^\bullet$ Let $(X, \mathcal{B}, \mu)$ be a probability space and let F be a real-valued random variable. Let $\nu = F_*(\mu)$ be the distribution of F. One defines the *moment generating function* ϕ for F as follows:

$$\phi(t) = \int_X exp(tF(x))\mu(dx)$$

where t is any real number. Verify that:

$$\phi(t) = \int_{\mathbf{R}} exp(ty)\nu(dy)$$

Note that $\phi(0) = 1$. Show that, for any positive integer k:

$$\phi^{(k)}(0) = m_k(F)$$

where, as usual, $m_k(F)$ is the k-th moment of F:

$$m_k(F) = \int_X F(x)^k \mu(dx) = \int_{\mathbf{R}} y^k \nu(dy)$$

Of course, we are making implicit assumptions that the foregoing integrals exist.

03$^\bullet$ Let $(X, \mathcal{B}, \mu)$ be a probability space and let F be a real-valued random variable for which the range is included in the set $\mathbf{Z}_0^+$ composed of all nonnegative integers. Let $\nu = F_*(\mu)$. Assume that there is a positive real number α such that:

$$\nu(\{n\}) = \frac{1}{n!}\alpha^n exp(-\alpha)$$

where n is any nonnegative integer. In this situation, one says that ν is a *poisson distribution* with parameter α. Let ϕ be the moment generating function for F. Show that:

$$\phi(t) = exp(\alpha(exp(t) - 1))$$

where t is any real number. Calculate the mean $m(F)$ and the variance $s^2(F)$ of F:

$$m(F) = m_1(F), \quad s^2(F) = m_2(F) - (m_1(F))^2$$