MATHEMATICS 331
ASSIGNMENT 9: Solution to Problem 04
Due: April 9, 2015

01° On $\mathbf{R}^3$, we have the conventional pdo geometry, defined by the following bilinear form:

$$\langle\!\langle \mathbf{x}, \mathbf{y} \rangle\!\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $\mathbf{x}$ and $\mathbf{y}$ are any members of $\mathbf{R}^3$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Convert the basis:

$$B_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for $\mathbf{R}^3$ to an orthonormal basis, causing minimal disturbance.

02° Let $\mathbf{V}$ be the linear space consisting of all polynomials h with real coefficients, having degree no greater than 3:

$$h(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Let $\mathbf{V}$ be supplied with a pdo geometry, as follows:

$$\langle\!\langle f, g \rangle\!\rangle = \int_{-1}^{1} f(x) g(x) dx$$

where f and g are any polynomials in $\mathbf{V}$. Introduce the following basis:

$$\mathcal{B}: \quad b_0, b_1, b_2, b_3$$

for $\mathbf{V}$, where:

$$b_j(x) = x^j \qquad (0 \leq j \leq 3, \; x \in \mathbf{R})$$

Convert $\mathcal{B}$ to an orthonormal basis for $\mathbf{V}$, causing minimal disturbance. Now let S be the linear mapping in $\mathbf{L}(\mathbf{V})$, defined by differentiation:

$$S(h) = h'$$

where h is any polynomial in $\mathbf{V}$. Describe the adjoint T of S. Is S self adjoint?

03° Let $\mathbf{V}'$ and $\mathbf{V}''$ be pdo geometries. Let S and T be a linear mappings in $\mathbf{L}(\mathbf{V}', \mathbf{V}'')$ and $\mathbf{L}(\mathbf{V}'', \mathbf{V}')$, respectively. Show that if S and T are adjoints of one another then the compositions TS and ST in $\mathbf{L}(\mathbf{V}')$ and $\mathbf{L}(\mathbf{V}'')$, respectively, are self adjoint.

04° Let $\mathbf{V}$ be a pdo geometry. Let P be a linear mapping in $\mathbf{L}(\mathbf{V})$ for which $PP = P$. Show that the conditions:

(1) $\mathbf{V} = ran(P) \perp ker(P)$

(2) P is self adjoint

are equivalent. Under the condition $PP = P$, we say that P is a *projection*. Sometimes, we say "orthogonal projection" rather than "self adjoint projection." Take special note of condition (1). It figures in both the Spectral Theorem and the Singular Value Decomposition. Verify that if P is a self adjoint projection then $Q = I - P$ is also a self adjoint projection, while:

$$ran(Q) = ker(P), \quad ker(Q) = ran(P)$$

[Let us note first that, for any Y in $ran(P)$, there is some X in $\mathbf{V}$ for which $Y = P(X)$, so that $P(Y) = P(P(X)) = P(X) = Y$. Now let us prove that (2) implies (1). To that end, we define $Q = I - P$. Clearly, Q is self adjoint, $P + Q = I$, $PQ = 0 = QP$, and $QQ = Q$. Hence, for any X in $\mathbf{V}$, we find that $X = I(X) = P(X) + Q(X)$. Obviously, $P(X)$ lies in $ran(P)$. Moreover, $Q(X)$ lies in $ker(P)$, since $PQ = 0$, and so, in turn, $\langle\!\langle P(X), Q(X) \rangle\!\rangle = \langle\!\langle X, P(Q(X)) \rangle\!\rangle = 0$. Finally, for any Y in $ran(P)$ and Z in $ker(P)$, if $Y + Z = 0$ then $\langle\!\langle Y, Z \rangle\!\rangle = \langle\!\langle P(Y), Z \rangle\!\rangle = \langle\!\langle Y, P(Z) \rangle\!\rangle = 0$. It follows that $0 = \langle\!\langle Y, Y + Z \rangle\!\rangle = \langle\!\langle Y, Y \rangle\!\rangle$, so that $Y = 0$, hence that $Z = 0$. We infer that $\mathbf{V} = ran(P) \perp ker(P)$. In turn, let us prove that (1) implies (2). To that end, et Y_1 and Y_2 be any members of $ran(P)$ and let Z_1 and Z_2 be any members of $ker(P)$. We obtain:

$$\langle\!\langle P(Y_1 + Z_1), Y_2 + Z_2 \rangle\!\rangle = \langle\!\langle Y_1, Y_2 \rangle\!\rangle = \langle\!\langle Y_1 + Z_1, P(Y_2 + Z_2) \rangle\!\rangle$$

We infer that S is self adjoint. The proof is complete.]