MATHEMATICS 331
ASSIGNMENT 6
Due: March 5, 2015

01° Let us engage in the computation of determinants. Let $\mathbf{V}$ be a finite dimensional linear space, having dimension n, and let L be a linear mapping carrying $\mathbf{V}$ to itself. Let us compute:

$$det(L)$$

To be explicit, let us assume that $n = 3$, so that $dim(\mathbf{V}) = 3$. Let us introduce a basis for $\mathbf{V}$:

$$\mathcal{B}: \quad B_1, B_2, B_3$$

As usual:

$$\begin{aligned} L(B_1) &= m_{11}B_1 + m_{21}B_2 + m_{31}B_3 \\ L(B_2) &= m_{12}B_1 + m_{22}B_2 + m_{32}B_3 \\ L(B_3) &= m_{13}B_1 + m_{23}B_2 + m_{33}B_3 \end{aligned}$$

and:

$$\bar{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

is the rectangular (indeed, square) array defined by L relative to the bases $\mathcal{B}$ and $\mathcal{B}$ for $\mathbf{V}$ and $\mathbf{V}$. We call for a proof that:

$$(*) \qquad det(L) = \sum_{\sigma} |\sigma| m_{1\sigma(1)} m_{2\sigma(2)} m_{3\sigma(3)} = \sum_{\tau} |\tau| m_{\tau(1)1} m_{\tau(2)2} m_{\tau(3)3}$$

There are six terms in each summation.

02° Both of the foregoing summations yield the sum of the following list of six numbers:

$$+m_{11}m_{22}m_{33} = +m_{11}m_{22}m_{33}$$

$$-m_{11}m_{23}m_{32} = -m_{11}m_{32}m_{23}$$

$$-m_{13}m_{22}m_{31} = -m_{31}m_{22}m_{13}$$

$$-m_{12}m_{21}m_{33} = -m_{21}m_{12}m_{33}$$

$$+m_{12}m_{23}m_{31} = +m_{31}m_{12}m_{23}$$

$$+m_{13}m_{21}m_{32} = +m_{21}m_{32}m_{13}$$

03° You might like to view the products in the foregoing list, in terms of the "neutral positions" of three rooks on a three by three chess board. In such a position, no one rook can attack either of the other two. Consequently, no two of the rooks can occupy the same row or the same column. Let us label the rooks by the positive integers 1, 2, and 3. Now every neutral position defines a permutation in $\mathbf{S}_3$, in two different ways, as follows:

$$\sigma(j) = k$$

iff the rook named j occupies the k-th column of the chess board. Just as well:

$$\tau(j) = k$$

iff the rook named j occupies the k-th row of the chess board. Obviously, σ and τ are inverse to one another. These permutations correspond to the following products in the foregoing list:

$$\begin{aligned} \sigma \ &\longrightarrow\ m_{1\sigma(1)}m_{2\sigma(2)}m_{3\sigma(3)} \\ \tau \ &\longrightarrow\ m_{\tau(1)1}m_{\tau(2)2}m_{\tau(3)3} \end{aligned}$$

Of course, the plus and minus signs attached to the products reflect the signs of the corresponding permutations. You should note that σ and τ have the same sign, that is, both are even or both are odd.

04° Now provide the proof of relation $(*)$. To that end, you might introduce the basis:

$$\mathcal{L}: \quad \Lambda_1, \Lambda_2, \Lambda_3$$

for $\mathbf{V}^*$, related to the basis $\mathcal{B}$ as follows:

$$\Lambda_j(B_k) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

Then you might form the member H of $\mathbf{L}^3(\mathbf{V})$:

$$H(X_1, X_2, X_3) = 6\,\Lambda_1(X_1)\Lambda_2(X_2)\Lambda_3(X_3)$$

and the basic member Δ of $\mathbf{\Lambda}^3(\mathbf{V})$, defined by anti-symmetrization of H:

$$\Delta = \mathcal{A}(H)$$

In point of fact, you may introduce any member Δ of $\mathbf{\Lambda}^3(\mathbf{V})$, so long as $\Delta \neq 0$. Finally, you must show that:

$$L^*(\Delta) = \delta\,\Delta$$

where:

$$\delta = \sum_{\sigma} |\sigma| m_{1\sigma(1)}m_{2\sigma(2)}m_{3\sigma(3)} = \sum_{\tau} |\tau| m_{\tau(1)1}m_{\tau(2)2}m_{\tau(3)3}$$

In fact, you need only show that:

$$L^*(\Delta)(B_1, B_2, B_3) = \delta\,\Delta(B_1, B_2, B_3)$$

Why?

05° Of course, $L^*(\Delta)$ is the pullback of Δ by L:

$$L^*(\Delta)(X_1, X_2, X_3) = \Delta(L(X_1), L(X_2), L(X_3))$$

In all the foregoing relations, X_1, X_2 and X_3 are arbitrary members of $\mathbf{V}$.

06° Let D be a member of $\mathbf{\Lambda}^3(\mathbf{F}^3)$ such that:

$$D(E_1, E_2, E_3) = 0$$

Show that $D = 0$.