MATHEMATICS 331
ASSIGNMENT 5
Due: February 26, 2015

01• Let n be a positive integer and let $\mathcal{N}$ be the set consisting of the first n positive integers:

$$\mathcal{N} = \{1, 2, 3, \ldots, n\}$$

Let $\mathbf{S}_n$ be the set of all bijections carrying the set $\mathcal{N}$ to itself. We refer to the members of $\mathbf{S}_n$ as *permutations*. For any members σ and τ, the composition:

$$\tau \cdot \sigma$$

is itself a bijection carrying $\mathcal{N}$ to itself. Under this operation of composition, $\mathbf{S}_n$ is a *group*. The identity mapping ϵ carrying $\mathcal{N}$ to itself serves as the identity element for $\mathbf{S}_n$:

$$\epsilon \cdot \sigma = \sigma = \sigma \cdot \epsilon$$

Of course, the operation is associative:

$$\upsilon \cdot (\tau \cdot \sigma) = (\upsilon \cdot \tau) \cdot \sigma$$

It is not commutative. Moreover, for every member σ of $\mathbf{S}_n$, there is a member τ of $\mathbf{S}_n$ such that:

$$\sigma \cdot \tau = \epsilon = \tau \cdot \sigma$$

Of course, τ is the mapping inverse to σ: $\tau = \sigma^{-1}$. Now let j and k be (positive) integers in $\mathcal{N}$ for which $j < k$. Let π be the permutation in $\mathbf{S}_n$ defined as follows:

$$\pi(\ell) = \begin{cases} \ell & \text{if } \ell \neq j \text{ and } \ell \neq k \\ k & \text{if } \ell = j \\ j & \text{if } \ell = k \end{cases}$$

We refer to π as a *transposition*. By a simple induction argument, one may prove that, for any σ in $\mathbf{S}_n$, there exist transpositions:

$$\pi_1, \pi_2, \ldots, \pi_r$$

such that:

$$\sigma = \pi_1 \cdot \pi_2 \cdot \cdots \cdot \pi_r$$

By the following two articles, one may prove that, for any two such presentations of σ:

$$\sigma = \pi'_1 \cdot \pi'_2 \cdot \cdots \cdot \pi'_p, \qquad \sigma = \pi''_1 \cdot \pi''_2 \cdot \cdots \cdot \pi''_q$$

the numbers p and q must have the same parity, which is to say that both p and q are even or both p and q are odd.

02° Let $\mathbf{A}$ be the set of all functions A of n variables, in the following form:

$$A(X_1, X_2, \ldots, X_n)$$

We imagine that the variables stand for arbitrary members of some hypothetical set $\mathbf{V}$. Let the group $\mathbf{S}_n$ act on the set $\mathbf{A}$ as follows:

$$(\sigma \cdot A)(X_1, X_2, \ldots, X_n) = A(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)})$$

where σ is any member of $\mathbf{S}_n$ and where A is any member of $\mathbf{A}$. Verify that, for any members σ and τ of $\mathbf{S}_n$ and for any member A of $\mathbf{A}$:

$$(\tau \cdot \sigma) \cdot A = \tau \cdot (\sigma \cdot A)$$

To do so, set $n = 6$. Interpret the foregoing definition in terms of the following notation:

$$\begin{aligned} X_{\sigma(1)} &= \sigma(X_1) \\ X_{\sigma(2)} &= \sigma(X_2) \\ X_{\sigma(3)} &= \sigma(X_3) \\ X_{\sigma(4)} &= \sigma(X_4) \\ X_{\sigma(5)} &= \sigma(X_5) \\ X_{\sigma(6)} &= \sigma(X_6) \end{aligned}$$

Note that σ does not change the given variables. It simply permutes them. In effect, the action of σ on $\mathcal{N}$ has migrated to a corresponding action on the variables. It is the same for τ and $\tau \cdot \sigma$.

03° Review the foregoing articles 01$^\bullet$ and 02°. Consider the function:

$$\Phi(X_1, X_2, \ldots, X_n) = \prod_{1 \le j < k \le n} (X_k - X_j)$$

Show that, for any transposition π:

$$\pi \cdot \Phi = -\Phi$$

Show that this fact proves the claim about parity at the end of the first article. To prove the fact, note that the transposition $\pi = (pq)$ will change the sign of Φ precisely $2b + 1$ times, where b is the number of integers between p and q. Of course, b might be 0. In any case, $2b + 1$ is odd.