MATHEMATICS 331
ASSIGNMENT 3
Due: February 12, 2015

01° Find two bases:

$$\mathcal{B}' : \; B'_1, B'_2, B'_3, B'_4; \qquad \mathcal{B}'' : \; B''_1, B''_2, B''_3, B''_4$$

for $\mathbf{R}^4$ such that:

$$B'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, B'_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, B''_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, B''_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

but which have no members in common.

02° Show that the sequence $\mathcal{C}$:

$$C_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, C_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, C_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, C_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, C_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, C_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

generates $\mathbf{R}^4$. By Reduction of $\mathcal{C}$, find a basis for $\mathbf{R}^4$.

03° Let $\mathbf{P}$ be the linear space composed of all polynomials of degree not greater than 8:

$$P(x) = \sum_{j=0}^{8} c_j x^j$$

where the c_j $(0 \leq j \leq 8)$ are any real numbers and where is a real variable. Let L be the linear mapping carrying $\mathbf{P}$ to itself, defined as follows:

$$L(P)(x) = \int_0^x P^{\circ\circ}(y)dy + 2P^{\circ\circ\circ}(x)$$

where P is any polynomial in $\mathbf{P}$ and where x is a real variable. Describe the linear subspaces $ker(L)$ and $ran(L)$ and find their dimensions.

04° Find a member B_4 of $\mathbf{R}^4$ such that:

$$\mathcal{B} : \; B_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}, B_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, B_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, B_4$$

is a basis for $\mathbf{R}^4$, or show that it cannot be done.

05° Let $\mathbf{V}$ be a linear space. Let $\mathcal{L}(\mathbf{V})$ be the set consisting of all linear mappings carrying $\mathbf{V}$ to itself. Of course, $\mathcal{L}(\mathbf{V})$ is an *algebra* under the familiar operations of addition, scalar multiplication, and multiplication:

$$\begin{aligned}(L' + L'')(X) &= L'(X) + L''(X)\\ (cL)(X) &= cL(X)\\ (L''L')(X) &= L''(L'(X))\end{aligned}$$

where L', L'', and L are any linear mappings in $\mathcal{L}(\mathbf{V})$, where c is any number in $\mathbf{F}$, and where X is any member of $\mathbf{V}$. Let L be a linear mapping in $\mathcal{L}(\mathbf{V})$. Show that if:

$$L^2 - L + I = 0$$

then L is invertible.

06° In context of the foregoing problem, show that if $dim(\mathbf{V}) = n$ then $dim(\mathcal{L}(\mathbf{V})) = n^2$. The first problem in the second assignment should prove helpful.