MATHEMATICS 331
ASSIGNMENT 2
Due: February 5, 2015

01° Let **M** be the set of all matrices (with real entries) which have 2 rows and 3 columns. Let **M** be supplied with the operations of addition and scalar multiplication. Note that **M** is a linear space. Describe the matrix which serves as the neutral matrix 0 for **M**. Show that the following sequence $\mathcal{B}$ of six members of **M** is a basis for **M**:

$$\mathcal{B}: \quad \begin{aligned} B_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ B_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ B_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ B_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ B_5 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ B_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

02° Let **P** be the set of all polynomial functions of the form:

$$P(x) = \sum_{j=0}^{5} c_j x^j$$

where c_0, c_1, c_2, c_3, c_4, and c_5 are any (real) numbers. Let **P** be supplied with the familiar operations of addition and scalar multiplication. Note that the following sequence $\mathcal{P}$ of six members of **P** is a basis for **P**:

$$\mathcal{P}: \quad \begin{aligned} P_0(x) &= x^0 = 1 \\ P_1(x) &= x^0 + x^1 = 1 + x \\ P_2(x) &= x^0 + x^1 + x^2 \\ P_3(x) &= x^0 + x^1 + x^2 + x^3 \\ P_4(x) &= x^0 + x^1 + x^2 + x^3 + x^4 \\ P_5(x) &= x^0 + x^1 + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

Let L be the mapping carrying $\mathbf{P}$ to itself, defined as follows:

$$L(P) = P^{\circ\circ}$$

where P is any member of $\mathbf{P}$ and where x is any (real) number. Of course, the domain and the codomain for L are the same. Let both of them be supplied with the basis $\mathcal{P}$. Find the matrix for L relative to $\mathcal{P}$ and $\mathcal{P}$.

03° In context of the foregoing two problems, describe a linear isomorphism L carrying $\mathbf{M}$ to $\mathbf{P}$. In fact, there are many. For the linear isomorphism which you have described, display the matrix relative to $\mathcal{B}$ and $\mathcal{P}$.

04° Let $\mathbf{V}$ be a linear space. Let $\mathcal{B}$ be a basis for $\mathbf{V}$ containing precisely two members:

$$\mathcal{B}: \quad B_1, B_2$$

Consequently, $\mathbf{V}$ is two dimensional. Let L' and L'' be linear mappings carrying $\mathbf{V}$ to $\mathbf{V}$ and let M' and M'' be the matrices for L' and L'', respectively, relative to $\mathcal{B}$ and $\mathcal{B}$:

$$L' \longleftrightarrow M' = \begin{pmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{pmatrix}, \quad L'' \longleftrightarrow M'' = \begin{pmatrix} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{pmatrix}$$

Let $L = L'' \cdot L'$ be the composition of L' and L'' and let M be the matrix for L relative to $\mathcal{B}$ and $\mathcal{B}$:

$$L \longleftrightarrow M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

By definition:

$$L(X) = L''(L'(X))$$

where X is any member of $\mathbf{V}$, and:

$$M''M' = \begin{pmatrix} m''_{11}m'_{11} + m''_{12}m'_{21} & m''_{11}m'_{12} + m''_{12}m'_{22} \\ m''_{21}m'_{11} + m''_{22}m'_{21} & m''_{21}m'_{12} + m''_{22}m'_{22} \end{pmatrix}$$

By EXPLICIT calculation, verify that:

$$M = M''M'$$

To do so, you must recognize that, by definition:

$$\begin{aligned} L'(B_1) &= m'_{11}B_1 + m'_{21}B_2 & L'(B_2) &= m'_{12}B_1 + m'_{22}B_2 \\ L''(B_1) &= m''_{11}B_1 + m''_{21}B_2, & L''(B_2) &= m''_{12}B_1 + m''_{22}B_2 \\ L(B_1) &= m_{11}B_1 + m_{21}B_2 & L(B_2) &= m_{12}B_1 + m_{22}B_2 \end{aligned}$$

05$^\bullet$ Study the following example. Let $\mathcal{E}'$ and $\mathcal{E}''$ be the standard bases for $\mathbf{R}^3$ and $\mathbf{R}^5$, respectively:

$$\mathcal{E}' : \quad E_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; E_2' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; E_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathcal{E}' : \quad E_1'' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; E_2'' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; E_3'' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; E_4'' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; E_5'' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Let Λ be a linear mapping carrying $\mathbf{R}^3$ to $\mathbf{R}^5$. By definition, the matrix M for Λ relative to $\mathcal{E}'$ and $\mathcal{E}''$ stands as follows:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \\ m_{51} & m_{52} & m_{53} \end{pmatrix}$$

where:

$$\Lambda(E_1') = m_{11}E_1'' + m_{21}E_2'' + m_{31}E_3'' + m_{41}E_4'' + m_{51}E_5'' = \begin{pmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \\ m_{51} \end{pmatrix}$$

$$\Lambda(E_2') = m_{12}E_1'' + m_{22}E_2'' + m_{32}E_3'' + m_{42}E_4'' + m_{52}E_5'' = \begin{pmatrix} m_{12} \\ m_{22} \\ m_{32} \\ m_{42} \\ m_{52} \end{pmatrix}$$

$$\Lambda(E_3') = m_{13}E_1'' + m_{23}E_2'' + m_{33}E_3'' + m_{43}E_4'' + m_{53}E_5'' = \begin{pmatrix} m_{13} \\ m_{23} \\ m_{33} \\ m_{43} \\ m_{53} \end{pmatrix}$$

In this way, Λ defines M. Just as well, M defines Λ. In fact, for any $\mathbf{x}$ in $\mathbf{R}^3$ and for any $\mathbf{y}$ in $\mathbf{R}^5$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

we find that:

$$\mathbf{y} = \Lambda(\mathbf{x}) \iff \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \\ m_{51} & m_{52} & m_{53} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

because:

$$\begin{aligned} \Lambda(\mathbf{x}) &= \Lambda(x_1 E'_1 + x_2 E'_2 + x_3 E'_3) \\ &= x_1 \Lambda(E'_1) + x_2 \Lambda(E'_2) + x_3 \Lambda(E'_3) \\ &= x_1 \begin{pmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \\ m_{51} \end{pmatrix} + x_2 \begin{pmatrix} m_{12} \\ m_{22} \\ m_{32} \\ m_{42} \\ m_{52} \end{pmatrix} + x_3 \begin{pmatrix} m_{13} \\ m_{23} \\ m_{33} \\ m_{43} \\ m_{53} \end{pmatrix} \\ &= \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \\ m_{51} & m_{52} & m_{53} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

Clearly, the correspondence between linear mappings Λ carrying $\mathbf{R}^3$ to $\mathbf{R}^5$ and matrices M having 5 rows and 3 columns is bijective. Moreover, composition of linear mappings corresponds to multiplication of matrices. Of course, one may replace the positive integers 3 and 5 by any positive integers p and q.