EXAMINATION
MATHEMATICS 322
Due: L306, High Noon, Wednesday, December 16, 2015
NO LIVING SOURCES

$01^\bullet$ Find all real valued functions x_1, x_2, x_3, and x_4 which are defined on $\mathbf{R}$ and which satisfy the following homogeneous linear ODE:

$$\begin{aligned} x_1^\circ &= x_1 + x_2 - x_3 \\ x_2^\circ &= x_2 + x_4 \\ x_3^\circ &= x_3 + x_4 \\ x_4^\circ &= x_4 \end{aligned}$$

$02^\bullet$ Find all real valued functions x which are defined on $\mathbf{R}$ and which satisfy the following linear (but inhomogeneous) ODE:

$$x^{\circ\circ}(t) - x^\circ(t) - 2x(t) = sin(2t) \qquad (t \in \mathbf{R})$$

$03^\bullet$ Consider the following (nonlinear) ODE:

$$(*) \qquad mx^{\circ\circ}(t) = -GMm\frac{1}{x(t)^2} \qquad (0 < x(t))$$

where G, m, and M are positive constants. Let ξ be the maximum solution for equation $(*)$, subject to the initial conditions:

$$\xi(0) = a, \;\; \xi^\circ(0) = b \qquad (0 < a,\, 0 < b)$$

Let J be the interval of definition for ξ:

$$J = (p, q) \qquad (-\infty \leq p < 0 < q \leq \infty)$$

Let h be the function defined as follows:

$$h(x, v) = \frac{1}{2}mv^2 - GMm\frac{1}{x} \qquad (0 < x,\, v \in \mathbf{R})$$

Show that the function:

$$h(\xi(t), \xi^\circ(t)) \qquad (t \in J)$$

is constant. Of course, its constant value must be:

$$\epsilon = \frac{1}{2}mb^2 - GMm\frac{1}{a}$$

Show that:

$$\epsilon < 0 \implies q < \infty \text{ and } \lim_{t \to q} \xi(t) = 0$$

$$0 < \epsilon \implies q = \infty \text{ and } \lim_{t \to \infty} \xi(t) = \infty$$

For the case in which $\epsilon = 0$, one refers to b as the Escape Speed from the position a:

$$b^2 = 2G\frac{M}{a}$$

Now let G be the Gravitational Constant, let M be the mass of a spherically symmetric central body, and let a be its radius. For what values of M/a would b exceed the speed of light c?

04$^\bullet$ One of the central studies in our course produced the solutions of the (homogeneous) Wave Equation in $\mathbf{R}^3$ by Spherical Means. By imitating the method, show that one can produce the solutions of the (homogeneous) Wave Equation in $\mathbf{R}^2$ by Circular Means or show that the procedure does not work.

05$^\bullet$ Let $\mathbf{B}$ be the open ball of radius 1 in $\mathbf{R}^3$, centered at $(0,0,0)$:

$$(x, y, z) \in \mathbf{B} \quad \text{iff} \quad x^2 + y^2 + z^2 < 1$$

Let $\mathbf{S}$ be the sphere of radius 1 in $\mathbf{R}^3$, centered at $(0,0,0)$:

$$(u, v, w) \in \mathbf{S} \quad \text{iff} \quad u^2 + v^2 + w^2 = 1$$

Let δ be a complex valued function defined and continuous on $\mathbf{S}$. Let γ be the complex valued function defined on $\mathbf{B}$ as follows:

$$\gamma(x, y, z)$$
$$= \frac{1}{4\pi} \int\!\!\int_{\mathbf{S}} \frac{1 - (x^2 + y^2 + z^2)}{[(x-u)^2 + (y-v)^2 + (z-w)^2]^{3/2}} \delta(u, v, w) cos\,\theta\, d\phi\, d\theta$$

where:

$$x^2 + y^2 + z^2 < 1 \quad \text{and} \quad (u, v, w) = (cos\,\theta\, cos\,\phi, cos\,\theta\, sin\,\phi, sin\,\theta)$$

Show that γ satisfies Laplace's Equation in $\mathbf{B}$:

$$(\circ) \qquad (\triangle\gamma)(x, y, z) = \gamma_{xx}(x, y, z) + \gamma_{yy}(x, y, z) + \gamma_{zz}(x, y, z) = 0$$

Try to show that, for each (u, v, w) in $\mathbf{S}$:

$$(\bullet) \qquad \lim_{(x,y,z) \to (u,v,w)} \gamma(x, y, z) = \delta(u, v, w)$$