**MATHEMATICS 322**
ASSIGNMENT 7
Due: October 28, 2015

**The Homogeneous Wave Equation**

Let $f$ and $g$ be complex valued functions defined on $\mathbf{R}^3$. We propose to solve the Homogeneous Wave Equation:

$$(\circ) \qquad \gamma_{tt}(t,x,y,z) - (\triangle\gamma)(t,x,y,z) = 0$$

subject to the Initial Conditions:

$$(\bullet) \qquad \gamma(0,x,y,z) = f(x,y,z), \qquad \gamma_t(0,x,y,z) = g(x,y,z)$$

Of course, $\gamma$ is the complex valued function defined on $\mathbf{R}^4$, required to be found. To be clear, we recall that:

$$(\triangle\gamma)(t,x,y,z) \equiv \gamma_{xx}(t,x,y,z) + \gamma_{yy}(t,x,y,z) + \gamma_{zz}(t,x,y,z)$$

**The Method of Fourier: Spherical Means**

We pass to the Fourier Transform of $\gamma$:

$$(\phi) \qquad \begin{aligned} \hat{\gamma}(t,u,v,w) &= \iiint_{\mathbf{R}^3} \gamma(t,x,y,z) e^{-i(ux+vy+wz)} m(dxdydz) \\ \gamma(t,x,y,z) &= \iiint_{\mathbf{R}^3} \hat{\gamma}(t,u,v,w) e^{+i(ux+vy+wz)} m(dudvdw) \end{aligned}$$

In the foregoing relations, we have adopted the following notational convention:

$$m(dudvdw) = \frac{1}{(2\pi)^{3/2}} dudvdw, \qquad m(dxdydz) = \frac{1}{(2\pi)^{3/2}} dxdydz$$

Clearly:

$$\begin{aligned} \gamma_{tt}(t,x,y,z) &= \iiint_{\mathbf{R}^3} \hat{\gamma}_{tt}(t,u,v,w) e^{+i(ux+vy+wz)} m(dudvdw) \\ (\triangle\gamma)(t,x,y,z) &= \iiint_{\mathbf{R}^3} -(u^2+v^2+w^2) \hat{\gamma}(t,u,v,w) e^{+i(ux+vy+wz)} m(dudvdw) \end{aligned}$$

We obtain the following reformulation of equations (∘) and (•):

$$(\circ) \qquad \hat{\gamma}_{tt}(t,u,v,w) + (u^2+v^2+w^2)\hat{\gamma}(t,u,v,w) = 0$$

$$(\bullet) \qquad \hat{\gamma}(0,u,v,w) = \hat{f}(u,v,w), \qquad \hat{\gamma}_t(0,u,v,w) = \hat{g}(u,v,w)$$

Now $\hat{\gamma}$ must take the form:

$$\begin{aligned} &\hat{\gamma}(t,u,v,w) \\ &\quad = \hat{f}(u,v,w)cos(\sqrt{u^2+v^2+w^2}\,t) \\ &\qquad\qquad + \hat{g}(u,v,w)\frac{1}{\sqrt{u^2+v^2+w^2}}\,sin(\sqrt{u^2+v^2+w^2}\,t) \end{aligned}$$

01• We need to recover $\gamma$ from $\hat{\gamma}$. To that end, let $h$ be a complex valued function defined on $\mathbf{R}^3$. Let $\nu_h$ be the complex valued function defined on $\mathbf{R}^4$ as follows:

$$(1) \qquad \nu_h(t,u,v,w) \equiv \hat{h}(u,v,w)\frac{1}{\sqrt{u^2+v^2+w^2}\,t}sin(\sqrt{u^2+v^2+w^2}\,t)$$

Verify that:

$$\hat{\gamma}(t,u,v,w) = \frac{\partial}{\partial t}t\,\nu_f(t,u,v,w) + t\,\nu_g(t,u,v,w)$$

Let $\mu_h$ be the complex valued function defined on $\mathbf{R}^4$ as follows:

$$(2) \qquad \mu_h(t,x,y,z) \equiv \iiint_{\mathbf{R}^3} \nu_h(t,u,v,w)e^{+i(ux+vy+wz)}m(dudvdw)$$

Verify that:

$$(*) \qquad \gamma(t,x,y,z) = \frac{\partial}{\partial t}t\,\mu_f(t,x,y,z) + t\,\mu_g(t,x,y,z)$$

02• To check that the foregoing solution of the Wave Equation meets the required initial conditions, we show that, by definition (1):

$$\begin{aligned} &t\nu_h(t,u,v,w)\big|_{t=0} = 0 \\ &\qquad \frac{\partial}{\partial t}t\nu_h(t,u,v,w)\big|_{t=0} = \hat{h}(u,v,w) \\ &\qquad\qquad \frac{\partial^2}{\partial t^2}t\nu_h(t,u,v,w)\big|_{t=0} = 0 \end{aligned}$$

03$^\bullet$ But we need to present $\mu_f$ and $\mu_g$ in a more perspicuous form. To that end, review a prior class discussion to obtain::

$$(3) \qquad \frac{1}{\sqrt{u^2+v^2+w^2}} sin(\sqrt{u^2+v^2+w^2}) = \frac{1}{4\pi} \int\int_\Sigma e^{+i(u\bar{x}+v\bar{y}+w\bar{z})} cos(\theta) d\phi d\theta$$

where $\Sigma$ is the unit sphere in $\mathbf{R}^3$ and where:

$$\begin{aligned} \bar{x} &= cos(\theta)cos(\phi) \\ \bar{y} &= cos(\theta)sin(\phi) \\ \bar{z} &= sin(\theta) \end{aligned}$$

04$^\bullet$ Clearly, for any positive number $t$:

$$\frac{1}{\sqrt{u^2+v^2+w^2}\,t} sin(\sqrt{u^2+v^2+w^2}\,t) = \frac{1}{4\pi t^2} \int\int_\Sigma e^{+i(ut\bar{x}+vt\bar{y}+wt\bar{z})} t^2 cos(\theta) d\phi d\theta$$

Show that:

$$\nu_h(t,u,v,w) = \hat{h}(u,v,w) \frac{1}{4\pi t^2} \int\int_\Sigma e^{+i(ut\bar{x}+vt\bar{y}+wt\bar{z})} t^2 cos(\theta) d\phi d\theta$$

so that:

$$(4) \qquad \begin{aligned} \mu_h(t,x,y,z) &= \int\int\int_{\mathbf{R}^3} \nu_h(t,u,v,w) e^{+i(ux+vy+wz)} m(dudvdw) \\ &= \frac{1}{4\pi t^2} \int\int_\Sigma h(x+t\bar{x}, y+t\bar{y}, z+t\bar{z}) t^2 cos(\theta) d\phi d\theta \end{aligned}$$

Clearly, $\mu_h(t,x,y,z)$ is the average value of $h$ over the sphere of radius $t$ centered at $(x,y,z)$.

05$^\bullet$ Obviously, the foregoing relation is sensible for any value of $t$. One refers to $\mu_h$ as the Spherical Mean defined by $h$. Now we can present the solution $\gamma$ of the Wave Equation in terms of Spherical Means, as follows:

$$(*) \qquad \begin{aligned} &\gamma(t,x,y,z) \\ &\quad = \frac{\partial}{\partial t} \frac{t}{4\pi t^2} \int\int_\Sigma f(x+t\bar{x}, y+t\bar{y}, z+t\bar{z}) t^2 cos(\theta) d\phi d\theta \\ &\qquad\qquad + \frac{t}{4\pi t^2} \int\int_\Sigma g(x+t\bar{x}, y+t\bar{y}, z+t\bar{z}) t^2 cos(\theta) d\phi d\theta \end{aligned}$$

06$^\bullet$ Apply relation $(*)$ to find the solutions to the Wave Equation subject to the following initial conditions:

$$f(x,y,z) = exp(-r^2); \quad g(x,y,z) = 0$$

$$f(x,y,z) = 0; \quad g(x,y,z) = exp(-r^2)$$

where $r^2 = x^2 + y^2 + z^2$.

07$^\bullet$ Study this problem for class discussion. Let $\epsilon$ be a (small) positive number. Let $s$ be a smooth real valued function defined on $\mathbf{R}^3$ such that $s(0,0,0) \neq 0$ and such that, for each $(x,y,z)$ in $\mathbf{R}^3$, if $\epsilon < r$ then $s(x,y,z) = 0$. Apply relation $(*)$ to obtain (in theory) the solutions $\gamma^\circ$ and $\gamma^\bullet$ to the Wave Equation, subject to the initial conditions:

$$(\circ) \qquad f(x,y,z) = s(x,y,z); \quad g(x,y,z) = 0 \qquad \Longrightarrow \qquad \gamma^\circ(t,x,y,z)$$

$$(\bullet) \qquad f(x,y,z) = 0; \quad g(x,y,z) = s(x,y,z) \qquad \Longrightarrow \qquad \gamma^\bullet(t,x,y,z)$$

Let $t$ be any positive number. Let $\delta^\circ$ and $\delta^\bullet$ be the functions defined in terms of $\gamma^\circ$ and $\gamma^\bullet$, as follows:

$$\delta^\circ(x,y,z) = \gamma^\circ(t,x,y,z); \qquad \delta^\bullet(x,y,z) = \gamma^\bullet(t,x,y,z)$$

Describe the *supports* of $\delta^\circ$ and $\delta^\bullet$. By definition, the supports are the smallest closed subsets $S^\circ$ and $S^\bullet$ of $\mathbf{R}^3$ in the complements of which the functions $\delta^\circ$ and $\delta^\bullet$ are, respectively, equal to 0. The object of this basically geometric exercise is to show the pattern of "propagation" of the small disturbance $s$ at time 0, under the two quite different initial conditions $(\circ)$ and $(\bullet)$.