MATHEMATICS 322
ASSIGNMENT 1
Due: September 9, 2015

A Basic Non-Autonomous Case

01$^\bullet$ Let I be an open interval in $\mathbf{R}$ and let a and b be functions defined on I. Let $V = I \times \mathbf{R}$. Let $\mathcal{F}$ be the mapping carrying V to $\mathbf{R}$, defined as follows:

$$\mathcal{F}(t, x) := -a(t)\,x + b(t) \qquad ((t, x) \in V)$$

Let s be a number in J and let w be a member of $\mathbf{R}$. Let γ be the maximum integral curve for $\mathcal{F}$:

$$(\circ *) \qquad \gamma^\circ(t) + a(t)\gamma(t) = b(t) \qquad (t \in I)$$

such that:

$$(\bullet *) \qquad \gamma(s) = w$$

Show that:

$$\gamma(t) = e^{-A(t)}B(t) \qquad (t \in I)$$

where:

$$A^\circ(t) = a(t), \;\; A(s) = 0; \qquad B^\circ(t) = e^{A(t)}b(t), \;\; B(s) = w$$

Why is the domain J for γ equal precisely to I? For the case in which $b = 0$, note that:

$$(\circ *) \qquad \gamma^\circ(t) + a(t)\gamma(t) = 0 \qquad (t \in I)$$

$$(\bullet *) \qquad \gamma(s) = w$$

and:

$$\gamma(t) = e^{-A(t)}w \qquad (t \in I)$$

Flows

02$^\bullet$ Let F be the mapping carrying $V = \mathbf{R}^1$ to $\mathbf{R}^1$, defined as follows:

$$F(x) = 1 + x^2 \qquad (x \in \mathbf{R})$$

One obtains the following ODE:

$$(\circ) \qquad x^\circ = 1 + x^2$$

Describe the flow domain Δ and the flow mapping γ for F. Start by introducing the mapping γ defined as follows:

$$\gamma(t) = tan(t) \qquad (-\frac{\pi}{2} < t < \frac{\pi}{2})$$

and by verifying that γ is the maximum integral curve for F passing through 0 at time 0.

03$^\bullet$ Let F be the mapping carrying $\mathbf{R}^2$ to $\mathbf{R}^2$, defined as follows:

$$F(x_1, x_2) = (-x_2, x_1) \qquad ((x_1, x_2) \in \mathbf{R}^2)$$

One obtains the following ODE:

$$(\circ) \qquad \begin{aligned} x_1^\circ &= -x_2 \\ x_2^\circ &= x_1 \end{aligned}$$

Describe the flow domain Δ and the flow mapping γ for F. Start by introducing the mapping γ defined as follows:

$$\gamma(t) = (cos(t), sin(t)) \qquad (t \in \mathbf{R})$$

and by verifying that γ is the maximum integral curve for F passing through $(1, 0)$ at time 0.

Angular Momentum

04$^\bullet$ Return to the Gravitational Equation of Newton in articles 33° and 34° of Chapter 1. Let M (the angular momentum per unit mass) be the mapping carrying $V = (\mathbf{R}^3 \backslash \{\mathbf{0}\}) \times \mathbf{R}^3$ to $\mathbf{R}^3$, defined as follows:

$$M(x, v) = x \times v \qquad ((x, v) \in V)$$

Let γ be an integral curve for F:

$$\gamma(t) = (x(t), v(t)) = (x(t), x^\circ(t)) \qquad (t \in J)$$

Show that the mapping:

$$M(x(t), x^\circ(t))$$

carrying J to $\mathbf{R}^3$ is constant.

Confinement

$05^{\bullet}$ Let F be the mapping carrying $\mathbf{R}^2$ to $\mathbf{R}^2$, defined as follows:

$$F(x_1, x_2) = (x_1^2 - x_2 - 1, x_1 + x_1 x_2) \qquad ((x_1, x_2) \in \mathbf{R}^2)$$

One obtains the following ODE:

$$(\circ) \qquad \begin{aligned} x_1^{\circ} &= x_1^2 - x_2 - 1 \\ x_2^{\circ} &= x_1 + x_1 x_2 \end{aligned}$$

Let C be the subset of $\mathbf{R}^2$ consisting of all points (w_1, w_2) for which $w_1^2 + w_2^2 = 1$. Of course, C is the unit circle in $\mathbf{R}^2$. Let γ be an integral curve for F:

$$\gamma(t) = (x_1(t), x_2(t)) \qquad (t \in J)$$

Show that either $\gamma(J) \subseteq C$ or $\gamma(J) \cap C = \emptyset$. To that end, let h be the function defined as follows:

$$h(x_1, x_2) = x_1^2 + x_2^2 \qquad ((x_1, x_2) \in \mathbf{R}^2)$$

Verify that:

$$(\nabla h)(x_1, x_2) \bullet F(x_1, x_2) = 2x_1(x_1^2 + x_2^2 - 1)$$

Show that if $\gamma(J) \cap C \neq \emptyset$ then $\gamma(J) \subseteq C$. Informally, one may say that the integral curves for F must lie entirely inside C, on C, or outside C. Note that there is just one critical point for F. In fact, $F(x_1, x_2) = (0, 0)$ iff $(x_1, x_2) = (0, -1)$. Conclude that:

$$\gamma(J) \subseteq C \implies \gamma(J) = \{(0, -1)\} \text{ or } \gamma(J) = C \backslash \{(0, -1)\}$$