MATHEMATICS 321
ASSIGNMENT 9: Solutions
Due: November 11, 2015

01• Let X be a measure space, supplied with a borel algebra $\mathcal{A}$ and a measure μ. Let:

$$E_1,\, E_2,\; \dots\;,\, E_n,\; \dots$$

be a sequence of sets in $\mathcal{A}$. For each x in X, let J_x be the set of all positive integers n such that $x \in E_n$. Let A be the subset of X consisting of all points x for which J_x is infinite:

$$A = \bigcap_{k=1}^{\infty} \bigcup_{\ell=k}^{\infty} E_\ell$$

Note that A is contained in $\mathcal{A}$. Show that if:

$$(*) \qquad \sum_{\ell=1}^{\infty} \mu(E_\ell) < \infty$$

then:

$$\mu(A) = 0$$

[Since the series $(*)$ converges, we have:

$$\lim_{k\to\infty} \sum_{\ell=k}^{\infty} \mu(E_\ell) = 0$$

In turn, for each k:

$$\begin{aligned} \mu(A) &\leq \mu(\bigcup_{\ell=k}^{\infty} E_\ell) \\ &\leq \sum_{\ell=k}^{\infty} \mu(E_\ell) \end{aligned}$$

By relation $(*)$, $\mu(A) = 0$.]

02• Let X_1 and X_2 be sets, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be borel algebras of subsets of X_1 and X_2, respectively, and let μ be a measure defined on $\mathcal{A}_1$. Let F be a borel mapping carrying X_1 to X_2. Let ν be the measure defined on $\mathcal{A}_2$ which assigns to each borel set B in $\mathcal{A}_2$ the following value:

$$\nu(B) \equiv \mu(F^{-1}(B))$$

Very often, we denote ν by $F_*(\mu)$. In turn, let g be a complex valued borel function defined on X_2. Let f be the complex valued (borel) function defined on X_1 which assigns to each member x of X_1 the following value:

$$f(x) = g(F(x))$$

Very often, we denote f by $F^*(g)$. Show that if g is integrable with respect to ν then f is integrable with respect to μ and:

$$(*) \qquad \int_{X_1} f(x)\mu(dx) = \int_{X_2} g(y)\nu(dy)$$

That is:

$$\int_{X_1} F^*(g) \cdot \mu = \int_{X_2} g \cdot F_*(\mu)$$

[We may as usual decompose g as follows:

$$g = u + iv = (u^+ - u^-) + i(v^+ - v^-)$$

where $0 \leq u^+$, and so forth. Consequently, we need only respond to the case in which $0 \leq g$. If g is a characteristic function then relation $(*)$ is obvious. If g is a simple function (that is, a (nonnegative) linear combination of characteristic functions) then relations $(*)$ follows by linearity of integration. For the general case, we need only introduce an increasing sequence of simple functions converging pointwise to g, then apply the Monotone Convergence Theorem.]