MATHEMATICS 321
ASSIGNMENT 5
Due: October 7, 2015

01• Let P_1, P_2, and P_3 be three distinct points in $\mathbf{R}^2$. Let F_1, F_2, and F_3 be the mappings carrying $\mathbf{R}^2$ to itself, defined as follows:

$$F_1(X) = \frac{1}{2}(X + P_1), \quad F_2(X) = \frac{1}{2}(X + P_2), \quad F_3(X) = \frac{1}{2}(X + P_3)$$

where X is any point in $\mathbf{R}^2$. Note that F_1, F_2, and F_3 are contraction mappings, with contraction constants having the common value 1/2. Let $\mathcal{F}$ be the mapping carrying $\mathcal{H}(\mathbf{R}^2)$ to itself, defined as follows:

$$\mathcal{F}(L) = F_1(L) \cup F_2(L) \cup F_3(L)$$

where L is any member of $\mathcal{H}(\mathbf{R}^2$. Show that $\mathcal{F}$ is a contraction mapping. What is the contraction constant for $\mathcal{F}$? Let T be the (closed) triangular area defined by P_1, P_2 and P_3. Draw a picture of the set:

$$K = \mathcal{F}^3(T)$$

in $\mathcal{H}(\mathbf{R}^2)$.

[Let δ be the hausdorff metric on $\mathcal{H}(\mathbf{R}^2)$. Let L and M be any sets in $\mathcal{H}(\mathbf{R}^2)$. Let τ be any positive number for which $\delta(L, M) < \tau$. It is precisely the same to say that $L \subseteq N_\tau(M)$ and $M \subseteq N_\tau(L)$. (Just to be absolutely clear, let us note that the foregoing logical equivalence in the definition of $\delta(L, M)$ depends upon the fact that L and M are compact. Why?) Obviously:

$$F_j(L) \subseteq N_{\tau/2}(F_j(M)) \text{ and } F_j(L) \subseteq N_{\tau/2}(F_j(M)) \quad (1 \leq j \leq 3)$$

so that:

$$\delta(F_j(L), F_j(M)) < \frac{\tau}{2}$$

It follows easily that:

$$\delta(\mathcal{F}(L), \mathcal{F}(M)) < \frac{\tau}{2}$$

Consequently:

$$\delta(\mathcal{F}(L), \mathcal{F}(M)) \leq \frac{1}{2}\,\delta(L, M)$$

]

$02^\bullet$ Let $\mathbf{R}$ be the set of all real numbers, supplied with the usual metric. Let f be a continuous complex valued function defined on $\mathbf{R}$. We say that f has *compact support* iff there is a compact subset K of $\mathbf{R}$ such that, for each number x in $\mathbf{R}\backslash K$, $f(x) = 0$. Of course, such a function must be bounded. Let $\mathbf{X}$ be the set of all continuous complex valued functions defined on $\mathbf{R}$, having compact support. Let d be the uniform metric on $\mathbf{X}$:

$$d(f_1, f_2) = \sup_{x\in\mathbf{R}} |f_1(x) - f_2(x)| \qquad (f_1,\ f_2 \in \mathbf{X})$$

Let Q be the mapping carrying $\mathbf{X}$ to itself, defined as follows:

$$Q(f)(x) = xf(x) \qquad (f \in \mathbf{X},\ x \in \mathbf{R})$$

Show that Q is continuous on $\mathbf{X}$ or show that it is not so.

[Let $\bar{0}$ be the constant function defined on $\mathbf{R}$ with constant value 0. Of course, $\bar{0}$ lies in $\mathbf{X}$. Obviously, $Q(\bar{0}) = \bar{0}$. Let n be any positive integer. Let f_n be a function in $\mathbf{X}$ such that $d(f_n, \bar{0}) \leq 1/n$ and such that $f_n(n) = 1/n$. Clearly, $1 \leq d(Q(f_n), Q(\bar{0}))$. Now, $f_n \longrightarrow \bar{0}$ but $Q(f_n) \not\longrightarrow \bar{0}$. Consequently, Q is not continuous at $\bar{0}$.]

$03^\bullet$ Let X be a metric space. We say that X satisfies Condition $C^\bullet$ iff, for any decreasing sequence:

$$\cdots \subseteq C_j \subseteq \ \cdots \subseteq C_3 \subseteq C_2 \subseteq C_1$$

of nonempty closed subsets of X, the intersection:

$$\bigcap_{j=1}^{\infty} C_j$$

is nonempty. Show that if X satisfies Condition $C^\bullet$ then X is compact.

[Let σ be a sequence in X. For each positive integer j, let C_j be the (nonempty closed) subset of X defined as follows:

$$C_j = clo(\{\sigma_j, \sigma_{j+1}, \sigma_{j+2},\ \ldots\})$$

By Condition $C^\bullet$, we may introduce a member w of the intersection of the foregoing sets:

$$w \in \bigcap_{j=1}^{\infty} C_j$$

Now, by induction (and by the characteristic property of the closure of a set), we may introduce a strictly increasing sequence:

$$j_1 < j_2 < j_3 < \ \cdots$$

of positive integers such that:

$$\sigma_{j_1} \in N_1(w),\ \sigma_{j_2} \in N_{1/2}(w),\ \sigma_{j_3} \in N_{1/3}(w),\ \sigma_{j_4} \in N_{1/4}(w),\ \ldots$$

In this way, we obtain a subsequence of σ which converges to w.]