MATHEMATICS 321
ASSIGNMENT 3
Due: September 23, 2015

01• Let $I \equiv [\,0, 1\,]$ be the closed unit interval in $\mathbf{R}$. Let F be a continuous mapping carrying I to itself. Show that there must be at least one number x in I such that $F(x) = x$.

02• Let X_1 and X_2 be metric spaces and let F be a mapping carrying X_1 to X_2. Let Γ be the graph of F, that is, let Γ be the subset of $X_1 \times X_2$ defined as follows:

$$\Gamma = \{(x_1, x_2) \in X_1 \times X_2 \, : \, x_2 = F(x_1)\}$$

Show that if F is continuous then X_1 and Γ are homeomorphic.

[We note first that the product space $X_1 \times X_2$ carries certain properties, by definition. In particular, let σ_1 and σ_2 be sequences in X_1 and X_2, respectively. Let σ be the corresponding sequence in $X_1 \times X_2$, of which σ_1 and σ_2 are the components:

$$\sigma = (\sigma_1, \sigma_2) : \qquad \sigma(j) = (\sigma_1(j), \sigma_2(j)) \qquad (j \in \mathbf{Z}^+)$$

By the definition of the product metric on $X_1 \times X_2$, we know that, for any u_1 in X_1 and for any u_2 in X_2:

$$\sigma_1 \longrightarrow u_1, \ \sigma_2 \longrightarrow u_2 \quad \text{iff} \qquad \sigma \longrightarrow (u_1, u_2)$$

Let us introduce the (bijective) mapping H carrying X_1 to Γ:

$$H(x) = (x, F(x)) \qquad (x \in X_1)$$

Clearly:

$$\sigma_1 \longrightarrow u_1 \quad \Longrightarrow \quad H \cdot \sigma_1 = (\sigma_1, F \cdot \sigma_1) \longrightarrow (u_1, F(u_1)) = H(u_1)$$

while:

$$(\sigma_1, \sigma_2) \longrightarrow (u_1, u_2) \quad \Longrightarrow \quad H^{-1} \cdot (\sigma_1, \sigma_2) = \sigma_1 \longrightarrow u_1 = H^{-1}(u_1, u_2)$$

It follows that H is a homeomorphism.]

03• Let X be a metric space, with metric d. One says that X is *connected* iff, for any subsets U and V of X, if U and V are open, if $U \cap V = \emptyset$, and if $U \cup V = X$ then $U = \emptyset$ or $V = \emptyset$. For instance, $\mathbf{R}^2$ (with the conventional metric) is connected. See the fourth problem in the first assignment. Again, let X be a metric space, with metric d. Let Y be a subset of X. Of course, both Y and $clo(Y)$ are themselves metric spaces, as one may restrict d to $Y \times Y$ and $clo(Y) \times clo(Y)$, respectively. Prove that if Y is connected then $clo(Y)$ is connected. Show by example that $clo(Y)$ may be connected while Y is not.

[Of course, we view Y and $clo(Y)$ as subspaces of X. As a consequence, the various open subsets of Y are the intersections with Y of the various open subsets of X. The same is true for $clo(Y)$. Let us assume that $clo(Y)$ is not connected. By this assumption, we may introduce open subsets U and V of X such that:

$$(1) \qquad U \cap clo(Y) \neq \emptyset, \quad V \cap clo(Y) \neq \emptyset$$

while:

$$(U \cap clo(Y)) \cap (V \cap clo(Y)) = \emptyset, \quad (U \cap clo(Y)) \cup (V \cap clo(Y)) = clo(Y)$$

It follows that:

$$(U \cap Y) \cap (V \cap Y) = \emptyset, \quad (U \cap Y) \cup (V \cap Y) = Y$$

because $Y \subseteq clo(Y)$. If $U \cap Y = \emptyset$ then $U \cap clo(Y)) = \emptyset$, which contradicts (1); hence, $U \cap Y \neq \emptyset$. Similarly, $V \cap Y \neq \emptyset$. Now we may infer that Y is not connected. The argument is complete.]