MATHEMATICS 321
ASSIGNMENT 11
Due: December 2, 2015

01$^\bullet$ Let $(X, \mathcal{A}, \pi)$ be a probability space. Let F be a random variable and $\mu = F_*(\pi)$ be the corresponding distribution. Specifically, let λ be a positive number and let μ be determined as follows:

$$\mu(\{n\}) = exp(-\lambda)\frac{1}{n!}\lambda^n \qquad (n = 0, 1, 2, \ldots)$$

Verify (again) that both the mean and the variance of μ equal λ. Let:

$$F_1, F_2, F_3, \ldots$$

be an independent sequence of random variables having common distribution μ. Let n be a positive integer. Let a and b be nonnegative numbers for which $a < b$. Let A be the set in $\mathcal{A}$ defined by the following condition:

$$x \in A \text{ iff } a \leq (F_1(x) + F_2(x) + \cdot + F_n(x)) \leq b$$

Apply the Central Limit Theorem to estimate $\mu(A)$. You should obtain a number of the form:

$$\frac{1}{\sqrt{2\pi}} \int_r^s exp(-\frac{1}{2}y^2)dy$$

where r and s depend upon a, b, and λ.

02$^\bullet$ Let λ be lebesgue measure on $\mathbf{R}^+$. Verify the relation:

$$\frac{1}{x} = \int_{(0,\infty)} e^{-xt}\lambda(dt)$$

where x is any positive number. Apply the Theorem of Fubini to prove that:

$$\lim_{a\to\infty} \int_{(0,a)} \frac{sin\, x}{x}\,\lambda(dx) = \frac{\pi}{2}$$

03$^\bullet$ Let F be a borel mapping carrying $\mathbf{R}$ to $\mathbf{R}$. Let Γ be the graph of F, which by definition consists of all points (x, y) in $\mathbf{R}^2$ for which $y = F(x)$. Show that Γ is a borel subset of $\mathbf{R}^2$. Show that:

$$(*) \qquad \int_{\mathbf{R}^2} ch_\Gamma(x, y)\,\lambda(dxdy) = 0$$

In the foregoing relation, ch_Γ is the characteristic function of Γ and λ is lebesgue measure on $\mathbf{R}^2$.

04• Let λ be lebesgue measure on $[0,1]$ and let μ be the counting measure on $[0,1]$. By definition, $\mu(E) = |E|$, where E is any subset of $[0,1]$. In the foregoing relation, $|E|$ stands for the number of members of E. In particular, $|E| = \infty$ if E is infinite. Let $[0,1] \times [0,1]$ be supplied with the corresponding product measure $\lambda \times \mu$. Let f be the function defined on $[0,1] \times [0,1]$, which assigns to each point (x, y) in $[0,1] \times [0,1]$ the value 0 if $x \neq y$ and the value 1 if $x = y$. Of course, the values of f are finite and nonnegative. Verify that f is borel. Compute the two iterated integrals for f. Note that one of them equals 0 while the other equals 1. Hence, the Theorem of Fubini does not apply in this case. Why?

05° Show that the Lebesgue Theory of Integration generalizes the Riemann Theory of Integration.