MATHEMATICS 321
ASSIGNMENT 10
Due: November 18, 2015

01$^\bullet$ Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f be a nonnegative real valued borel function defined on X. Let ν be the function defined on $\mathcal{A}$ as follows:

$$\nu(A) = \int_A f(x)\mu(dx) \equiv \int_X \chi_A(x) f(x)\mu(dx)$$

where A is any set in $\mathcal{A}$. Show that ν is a measure. The following notation proves useful:

$$\nu = f \cdot \mu, \qquad \frac{d\nu}{d\mu} = f$$

In turn, let g be a complex valued borel function defined on X. Show that if g is integrable with respect to ν then gf is integrable with respect to μ and:

$$\int_X g(x)\nu(dx) = \int_X g(x)f(x)\mu(dx)$$

02$^\bullet$ Let λ be the lebesgue measure defined on the borel subsets of $\mathbf{R}$. Let h be a nonnegative real valued borel (indeed, continuous) function defined on $\mathbf{R}$, as follows:

$$h(y) = \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}y^2)$$

where y is any real number. Let ρ be the corresponding measure defined on the borel subsets of $\mathbf{R}$, in the manner described in the preceding problem:

$$\rho(B) = \int_B h(y)\lambda(dy)$$

where B is any borel subset of $\mathbf{R}$. ($\mathbf{R}$)Now ρ is a probability measure. That is, $\rho(\mathbf{R}) = 1$. Why? Let $\hat{\rho}$ be the corresponding characteristic function:

$$\hat{\rho}(t) = \int_{\mathbf{R}} exp(ity)\rho(dy)$$

where t is any real number. In the lectures, we will prove that:

$$(\bullet) \qquad \hat{\rho}(t) = exp(-\frac{1}{2}t^2)$$

where t is any real number. Use the foregoing relation to show that the mean m of ρ is 0 and the standard deviation s is 1.

03$^\bullet$ Prove the foregoing relation ($\bullet$) yourself. To do so, you might show, by differentiation, that:

$$\frac{\hat{\rho}(t)}{exp(-\frac{1}{2}t^2)} = 1$$

where t is any real number.

04° Let $(X, \mathcal{A}, \mu)$ be a measure space, for which the measure of X is finite:

$$\mu(X) < \infty$$

Let g be a bounded complex valued borel function defined on X. Show that g is a integrable.