MATHEMATICS 321
ASSIGNMENT 8
Due: November 04, 2015

01$^\bullet$ Let $(X, \mathcal{A}, \mu)$ be a measure space, with borel algebra $\mathcal{A}$ and measure μ defined on $\mathcal{A}$. Let:

$$B_1 \subseteq B_2 \subseteq B_3 \ \cdots$$

be an increasing sequence of sets in $\mathcal{A}$ and let B be the union of them:

$$B = \bigcup_{n=1}^{\infty} B_n$$

Show that:

$$\mu(B) = \lim_{n\to\infty} \mu(B_n)$$

[HINT. Note that:

$$B = B_1 \cup (B_2 \backslash B_1) \cup (B_3 \backslash (B_1 \cup B_2)) \cup \cdots \qquad]$$

02$^\bullet$ Let $(X, \mathcal{A}, \mu)$ be a measure space, with borel algebra $\mathcal{A}$ and measure μ defined on $\mathcal{A}$. Let f be a nonnegative extended real valued borel function defined on X. Let B and C be the sets in $\mathcal{A}$ defined as follows:

$$B = \{x \in X : 0 < f(x)\}, \ \ C = \{x \in X : f(x) = \infty\}$$

Show that:

$$\int_X f(x)\mu(dx) = 0 \Longrightarrow \mu(B) = 0, \ \ \int_X f(x)\mu(dx) < \infty \Longrightarrow \mu(C) = 0$$

[HINT. For each positive integer n, let D_n be the set in $\mathcal{A}$ defined as follows:

$$D_n = \{x \in X : \frac{1}{n} < f(x)\}$$

and note that:

$$B = \bigcup_{n=1}^{\infty} D_n, \ \ n\mu(C) \leq \int_X f(x)\mu(dx)$$

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03• Let $(X, \mathcal{A}, \mu)$ be a measure space, with borel algebra $\mathcal{A}$ and measure μ defined on $\mathcal{A}$. Let A and B be borel sets in $\mathcal{A}$ for which $A \cap B = \emptyset$, $0 < \mu(A)$, and $0 < \mu(B)$. Consider the sequence of measurable functions defined on X as follows:

$$f_n = \begin{cases} \chi_A & \text{if } n \text{ is odd} \\ \chi_B & \text{if } n \text{ is even} \end{cases}$$

Of course, χ_A and χ_B are the characteristic functions of A and B, respectively. Show that:

$$\int_X (\liminf_{n\to\infty} f_n) \cdot \mu \, < \, \liminf_{n\to\infty} \int_X f_n \cdot \mu$$

For convenience of expression, we have abbreviated the notation for integrals as follows:

$$\text{not} \int_X h \cdot \mu$$

04• Let us supply the set $\mathbf{R}$ with the borel algebra $\mathcal{X}$ consisting of all (!) subsets of $\mathbf{R}$. Let a be any positive real number. We are pleased to introduce a measure μ, defined on $\mathcal{X}$ in the following manner. For each set X in $\mathcal{X}$:

$$\mu(X) = e^{-a} \sum_{k \in \mathbf{N} \cap X} \frac{1}{k!} a^k$$

Note that $\mu(\mathbf{R} \backslash \mathbf{N}) = 0$ and $\mu(\mathbf{N}) = 1$, so that $\mu(\mathbf{R}) = 1$. Calculate:

$$m = \int_{\mathbf{R}} x \mu(dx)$$