Uniform Boundedness

01° Let $z = x + iy$ be a complex number for which the real part x is positive. Let us form the sequence:

$$(\bullet) \qquad w_0 = z,\ w_1 = z + \frac{1}{w_0},\ w_2 = z + \frac{1}{w_1},\ \ldots\ ,\ w_{k+1} = z + \frac{1}{w_k},\ \ldots$$

By induction, it is plain that, for each nonnegative integer k, the real parts of w_k and $1/w_k$ are positive. Hence:

$$(1) \qquad x \leq |w_k|$$

It follows that:

$$|w_{k+1}| = |z + \frac{1}{w_k}| \leq |z| + \frac{1}{|w_k|} \leq |z| + \frac{1}{x}$$

Hence:

$$(2) \qquad x \leq |w_k| \leq |z| + \frac{1}{x}$$

02° Now let K be a compact set of complex numbers such that, for each z in K, the real part x of z is positive. Let b be a positive number such that, for each z in K:

$$|z| + \frac{1}{x} \leq b$$

By the foregoing observations, it is plain that, for each z in K, the sequence $(\bullet)$ defined by z is bounded by b.

03° Obviously, if the sequence $(\bullet)$ is convergent then the limit w must satisfy the relation:

$$(\circ) \qquad w^2 - zw - 1 = 0$$

Of course, w must then be the zero for which the real part is positive. By the "subsubsequence" argument, the sequence $(\bullet)$ must in fact converge to w.

04° In context of Complex Analysis, we infer that sequence of analytic functions of z, defined by $(\bullet)$ in the right half plane, converges uniformly on compact sets to the limit defined by $(\circ)$.