MATHEMATICS 311
EXAMINATION
Due: Wednesday, May 13, 2015, NOON, Library 306

01• Let f be the complex valued function defined as follows:

$$f(z) = tan(\frac{1}{2i}log(\frac{1+iz}{1-iz}))$$

Describe the "natural" domain Ω for f. Show that:

$$f(z) = z \qquad (z \in \Omega)$$

02• Evaluate the integral:

$$\int_0^{2\pi} \frac{1}{a + b\,sin\,\theta} d\theta$$

where $a \in \mathbf{R}$, $b \in \mathbf{R}$, and $0 < |b| < a$.

03• Evaluate the contour integrals:

$$\int_\Gamma \frac{z\,exp(z)}{z + 2i} dz, \quad \int_\Delta \frac{z\,exp(z)}{z + 2i} dz$$

where:

$$\Gamma(t) = exp(it), \;\; \Delta(t) = 3\,exp(it), \;\; 0 \leq t \leq 2\pi$$

04• Let f be the complex valued function defined as follows:

$$f(z) = \frac{1}{(z-2)z(z+1)}$$

where $1 < |z| < 2$. Find the Laurent Expansion for f in the annulus on which it is defined.

05• Let f be a complex valued function defined and analytic on the entire complex plane $\mathbf{C}$. For each positive real number r, let:

$$M(r) = \max_{|z|=r} |f(z)|$$

Show that, for any positive real numbers r' and r'':

$$r' < r'' \Longrightarrow M(r') < M(r'')$$

06• Determine the number of complex numbers ζ for which $1 < |\zeta| < 2$ and:

$$\zeta^4 - 6\zeta + 3 = 0$$

07• Let Ω be the region in $\mathbf{C}$ defined as follows:

$$z \in \Omega \Longleftrightarrow [(0 < x) \text{ and } (x \leq 1 \Longrightarrow y \neq 0)] \qquad (z = x + iy)$$

Let f be the complex valued function defined on Ω as follows:

$$f(z) = i\sqrt{z^2 - 1} \qquad (z \in \Omega)$$

Confirm that f is analytic. Describe the range of f. Let u and v be the real and imaginary parts of f:

$$f(z) = w = u(x, y) + iv(x, y)$$

Sketch the level sets for u and v:

$$u(x, y) = a, \;\; v(x, y) = b$$

08• Let Ω be a region in $\mathbf{C}$ of the following form:

$$\Omega = \Omega^+ \cup J \cup \Omega^-$$

where Ω^+ is a region in $\mathbf{C}$ such that:

$$z \in \Omega^+ \Longrightarrow 0 < y \qquad (\text{where } z = x + iy)$$

where Ω^- is the region in $\mathbf{C}$ conjugate to Ω^+:

$$z \in \Omega^- \Longleftrightarrow \bar{z} \in \Omega^+$$

and where J be an open interval in $\mathbf{R}$. (Review the definition of a region.) Let f be a complex valued function defined and analytic on Ω^+ such that, for each (real) number u in J:

$$\lim_{z \to u} f(z) = 0$$

Show that, for each (complex) number z in Ω^+, $f(z) = 0$. To that end, introduce the complex valued function ϕ, defined on Ω as follows:

$$z \in \Omega \Longrightarrow \phi(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \\ 0 & \text{if } z \in J \\ \overline{f(\bar{z})} & \text{if } z \in \Omega^- \end{cases}$$

Show that ϕ is analytic. Finish the argument.

09• Find all solutions of the following equation:

$$f''(z) + zf(z) = 0$$

To that end, consider functions defined by power series.