COMPLEX DIFFERENTIATION

01° We identify $\mathbf{C}$ with $\mathbf{R}^2$, subject to the following notation:

$$z = x + iy = \begin{pmatrix} x \\ y \end{pmatrix}$$

Let W be a region in $\mathbf{C}$ and let f be a mapping carrying W to $\mathbf{C}$:

$$f(z) = w = u + iv = \begin{pmatrix} u \\ v \end{pmatrix} \qquad (z \in W)$$

By the foregoing identification, we may regard f as a mapping carrying W to $\mathbf{R}^2$:

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} u \\ v \end{pmatrix}$$

02° Let z_o be a member of W:

$$z_o = x_o + iy_o = \begin{pmatrix} x_o \\ y_o \end{pmatrix}$$

One says that f is *analytic* at z_o iff there is a member c of $\mathbf{C}$:

$$c = a + ib = \begin{pmatrix} a \\ b \end{pmatrix}$$

such that:

(1) $$\lim_{z \to z_o} \frac{1}{z - z_o}(f(z) - f(z_o) - c(z - z_o)) = 0$$

One says that f is *totally differentiable* at z_o iff there is a two by two matrix M:

$$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

having real entries such that:

(2) $$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_o \\ y_o \end{pmatrix}} \frac{1}{\| \begin{pmatrix} x - x_o \\ y - y_o \end{pmatrix} \|} \| f(\begin{pmatrix} x \\ y \end{pmatrix}) - f(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) - M \begin{pmatrix} x - x_o \\ y - y_o \end{pmatrix} \| = 0$$

In the latter case, it might (but may not) happen that $s = p$ and $r = -q$:

$$M = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \tag{3}$$

03° Given a member c of $\mathbf{C}$:

$$c = a + ib \tag{4}$$

we may introduce the following two by two matrix M:

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{5}$$

having real entries. Clearly:

$$cz = (ax - by) + i(bx + ay) = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$$

04° Now it is plain that (1) holds iff (2) and (3) hold, where c and M are linked by (4) and (5). We conclude that f is analytic at z_o iff f is totally differentiable at z_o and the following relations hold:

$$\frac{\partial u}{\partial x}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) = \frac{\partial v}{\partial y}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}), \qquad \frac{\partial u}{\partial y}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) = -\frac{\partial v}{\partial x}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) \tag{CR}$$

One calls these relations the Cauchy/Riemann Equations. Obviously:

$$f'(z_o) = \frac{\partial u}{\partial x}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) + i\frac{\partial v}{\partial x}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) = \frac{\partial v}{\partial y}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) - i\frac{\partial u}{\partial y}(\begin{pmatrix} x_o \\ y_o \end{pmatrix}) \tag{6}$$

05° Informally, we write the foregoing relations as follows:

$$u_x = v_y, \quad u_y = -v_x \tag{CR}$$

and:

$$f' = u_x + i\,v_x = v_y - i\,u_y \tag{7}$$