MATHEMATICS 311
ASSIGNMENT 8
Due: April 8, 2015

01° Let Ω be a region in $\mathbf{C}$. Let g be a function defined and analytic on Ω. Let:

$$f_1,\ f_2,\ f_3,\ \dots\ ,f_j,\ \dots$$

be a sequence of functions defined and analytic on Ω, which converges uniformly to g on compact subsets of Ω. We mean to say that:

$$(\forall K \subseteq \Omega)(\forall \epsilon \in \mathbf{R}^+)(\exists n \in \mathbf{Z}^+)(\forall j \in \mathbf{Z}^+) \Big[n \leq j \Longrightarrow (\forall z \in \Omega)[\, |f_j(z) - g(z)| \leq \epsilon\,]\Big]$$

That is, for each compact subset K of Ω and for any positive real number ϵ, there is some positive integer n such that, for any positive integer j, if $n \leq j$ then, for any member z of K:

$$|f_j(z) - g(z)| \leq \epsilon$$

Assume that, for each positive integer j, f_j is injective. We inquire whether or not g must be injective as well. Show by example that, in fact, g might be constant. Assume in turn that g is not constant. Prove that g must be injective.

02° Let $z = x + iy$ be a complex number for which $0 < x < 1$. Show that:

$$\int_0^\infty \frac{t^{-z}}{1+t}dt = \frac{\pi}{sin(\pi z)}$$

In the lectures, we will describe a contour integral by which the calculation can be made. The result will figure in our discussion of the Gamma Function.

03° Let Ω be the region in $\mathbf{C}$ defined by the conditions:

$$z = x + iy \in \Omega \quad \text{iff} \quad 0 < y,\ 1 < |z|$$

Let F be the analytic mapping carrying Ω to $\mathbf{C}$, defined as follows:

$$u + iv = w = F(z) = \frac{1}{z} + z$$

Describe $F(\Omega)$. Sketch the curves:

$$u(x, y) = c,\ \ v(x, y) = d$$

where c and d are various real numbers. Why do the curves appear to cross at right angles?

04° Let $\mathbf{\Delta}$ be the open unit disk in $\mathbf{C}$ centered at 0. Let f be the function defined on $\mathbf{\Delta}$ as follows:

$$f(z) = \frac{z}{(1-z)^2} \qquad (z \in \mathbf{\Delta})$$

Show that f is injective. Describe $f(\mathbf{\Delta})$.

05• For any mapping F carrying the right half-plane $\mathbf{E}$ to itself, let F^* be the mapping defined in terms of F as follows:

$$F^*(z) = z + \frac{1}{F(z)} \qquad (z \in \mathbf{E})$$

Show that F^* also carries $\mathbf{E}$ to itself. Now let G be the mapping (carrying $\mathbf{E}$ to itself) defined as follows:

$$G(z) = z \qquad (z \in \mathbf{E})$$

Form the sequence of mappings:

$$G, G^*, G^{**}, G^{***}, G^{****}, \ \ldots$$

Show that the sequence converges uniformly on compact subsets of $\mathbf{E}$. What is the limit function? We will return to this problem, repeatedly, until we solve it.