MATHEMATICS 311
ASSIGNMENT 7
Due: March 18, 2015

01° Let $\mathbf{\Delta}$ be the open disk in $\mathbf{C}$ centered at 0 with radius 1. Let $\mathbf{D}$ be the closure of $\mathbf{\Delta}$. Let f be a complex-valued function defined on $\mathbf{D}$, continuous on $\mathbf{D}$, and analytic on $\mathbf{\Delta}$. Assume that there is a positive real number ϵ such that:

$$0 \leq \theta \leq \epsilon \Longrightarrow f(e^{i\theta}) = 0$$

Show that f must be constantly 0 on $\mathbf{D}$.

02° Let n be an integer for which $2 < n$ and let f be the polynomial defined as follows:

$$f(z) = z^n - \frac{1}{4}(1 + z + z^2) \qquad (z \in \mathbf{C})$$

Show that the zeros of f lie in the unit disk $\mathbf{\Delta}$.

03° For any complex numbers ζ_1 and ζ_2, let us write:

$$\zeta_1 \bullet \zeta_2 = \Re(\zeta_1 \bar{\zeta}_2) = \Re(\zeta_1)\Re(\zeta_2) + \Im(\zeta_1)\Im(\zeta_2)$$

For each z in the unit disk $\mathbf{\Delta}$ and for any complex numbers ζ_1 and ζ_2, let us write:

$$\langle\!\langle\, \zeta_1, \zeta_2 \,\rangle\!\rangle_z = (\frac{2}{1 - |z|^2})^2\, \zeta_1 \bullet \zeta_2$$

Now let H be an automorphism of $\mathbf{\Delta}$:

$$H(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \qquad (z \in \mathbf{\Delta})$$

where $|\alpha|^2 - |\beta|^2 = 1$. Let z be any member of $\mathbf{\Delta}$ and let ζ_1 and ζ_2 be any complex numbers. Let:

$$w = H(z),\ \eta_1 = H'(z)\zeta_1,\ \eta_2 = H'(z)\zeta_2$$

Show that:

$$\langle\!\langle\, \eta_1, \eta_2 \,\rangle\!\rangle_w = \langle\!\langle\, \zeta_1, \zeta_2 \,\rangle\!\rangle_z$$

[If you like, you may restrict your attention to the case in which:

$$\alpha = cosh(\theta),\ \beta = sinh(\theta)$$

where θ is any real number.] Conclude that H is an *isometry* for the *metric structure* $\langle\!\langle\, \cdot, \cdot \,\rangle\!\rangle$ on $\mathbf{\Delta}$.