MATHEMATICS 311
ASSIGNMENT 5
Due: March 4, 2015

Let Ω be a region in $\mathbf{C}$, let S be a finite subset of Ω, let f be a function defined and analytic on $\Omega\backslash S$, and let Γ be a closed chain in $\Omega\backslash S$ such that Γ is homologous to 0 in Ω. The Residue Theorem states that:

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{w\in S} W(\Gamma,w)Res(f,w)$$

By $W(\Gamma,w)$, we mean the winding number of Γ relative to w. By $Res(f,w)$, we mean the residue of f at w.

01° Let γ be the (simple closed) path in $\mathbf{C}$ which traces ccw the circle centered at 0 with radius 2. Calculate:

$$\int_\gamma \frac{1}{z^2-1}dz$$

02° Let γ be the (simple closed) path in $\mathbf{C}$ which traces ccw the circle in centered at 0 with radius 7. Calculate:

$$\int_\gamma \frac{1+z}{1-cos(z)}dz$$

03° Let γ be any simple closed path in $\mathbf{C}\backslash\{0\}$. Calculate:

$$\int_\gamma \frac{exp(-z^2)}{z^2}dz$$

04° Calculate:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx$$

05° Calculate:

$$\int_0^{2\pi} exp(exp(it))dt$$

06° By a polynomial in the real variables x and y, we mean a complex valued function of the following form:

$$s(x,y) = \sum_{\ell=0}^{n} \Big[\sum_{0 \le j, 0 \le k, j+k=\ell} c_{jk} x^j y^k \Big]$$

where the various coefficients c_{jk} are complex numbers. We presume that the degree of s is n, which is to say that there is at least one coefficient c_{jk} for which $j + k = n$ and $c_{jk} \neq 0$. By a rational function in the real variables x and y, we mean a ratio of two polynomials, let them be p and q, in x and y:

$$r(x,y) = \frac{p(x,y)}{q(x,y)}$$

For such a function, we may define a corresponding complex valued function of a complex variable z:

$$f(z) = \frac{g(z)}{izh(z)}$$

where:

$$g(z) = p\big(\,(1/2)(z + (1/z)), (1/2i)(z - (1/z))\,\big)$$
$$h(z) = q\big(\,(1/2)(z + (1/z)), (1/2i)(z - (1/z))\,\big)$$

Verify that f is meromorphic on $\mathbf{C}$. Let P be the set of poles of f in $\mathbf{C}$. In turn, let Δ be the open unit disk in $\mathbf{C}$ centered at 0:

$$z \in \Delta \Longleftrightarrow |z| < 1$$

and let Γ be the boundary of Δ, that is, the unit circle in $\mathbf{C}$ centered at 0 with radius 1:

$$w \in \Gamma \Longleftrightarrow |w| = 1$$

Now assume that, for any real numbers u and v, if $u^2 + v^2 = 1$ then $q(u,v) \neq 0$. As a consequence, verify that, for each w in Γ, f is analytic at w. Finally, let $S = P \cap \Delta$. Verify that S is finite. Now apply the Residue Theorem to show that:

$$\frac{1}{2\pi i} \int_0^{2\pi} r(cos\theta, sin\theta) d\theta = \sum_{w \in S} Res(f, w)$$

Start by noting that:

$$r(cos\theta, sin\theta) = ie^{i\theta} f(e^{i\theta})$$

Then compute:

$$\frac{1}{2\pi i} \int_{\Gamma} f(z) dz$$

07° Let a be a real number for which $0 < a$ but $a \neq 1$. Show that:

$$\int_0^{2\pi} \frac{1}{1+a^2-2acos\theta} d\theta = \begin{cases} 2\pi/(1-a^2) & \text{if } a < 1 \\ 2\pi/(a^2-1) & \text{if } 1 < a \end{cases}$$

08° Let n be a positive integer. Show that:

$$\int_{-\pi}^{\pi} cos^{2n}\theta d\theta = \frac{2\pi}{4^n}\binom{2n}{n}$$