MATHEMATICS 311
ASSIGNMENT 3
Due: February 18, 2015

01$^\bullet$ The *Bernoulli Numbers* are defined by the following relation:

$$\frac{z}{exp(z) - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$$

Find B_0, B_1, B_2, B_3, and B_4. Show that for any integer k, if $3 \leq k$ and if k is odd then $B_k = 0$.

02$^\bullet$ The *Fibonacci Numbers* are defined by the following relation:

$$\frac{z}{1 - z - z^2} = \sum_{j=0}^{\infty} F_j z^j$$

Find the real numbers a and b for which:

$$1 - z - z^2 = (1 - az)(1 - bz), \quad a < 0 < b$$

Show that:

$$F_j = \frac{b^j - a^j}{\sqrt{5}} \qquad (j = 0, 1, 2, 3, \ldots)$$

To that end, first find the real numbers A and B such that:

$$\frac{z}{1 - z - z^2} = \frac{A}{1 - az} + \frac{B}{1 - bz}$$

Find the radius of convergence for the power series.

03$^\bullet$ Let z be any complex number and let r be any positive real number. Let $C_r(z)$ stand for the circle having center z and radius r:

$$w \in C_r(z) \quad \text{iff} \quad |w - z| = r$$

and let $D_r(z)$ stand for the open disk having center z and radius r:

$$w \in D_r(z) \quad \text{iff} \quad |w - z| < r$$

Let $\gamma_r(z)$ be the curve, parametrized by arclength, which traverses $C_r(z)$ once ccw:

$$\gamma_r(z)(s) = z + r exp(i\frac{1}{r}s), \qquad (0 \leq s \leq 2\pi r)$$

Let Ω be a region in $\mathbf{C}$ which includes:

$$clo(D_r(z)) = D_r(z) \cup C_r(z)$$

and let f be a function defined and analytic on Ω. Apply the Cauchy Integral Formula, for circular curves, to show that:

$$f(z) = \frac{1}{2\pi r}\int_0^{2\pi r} f(z + rexp(i\frac{1}{r}s))ds$$

Hence, $f(z)$ is the average of the values of f on $C_r(z)$. Apply polar coordinates to show that:

$$f(z) = \frac{1}{\pi r^2}\int\int_{D_z(r)} f(x+iy)dxdy$$

Hence, $f(z)$ is the average of the values of f on $D_r(z)$.

04• Let r be a positive real number for which $0 < r < 1$. Let f be the function defined as follows:

$$f(z) = \frac{1}{1-rz} \qquad (|z| < \frac{1}{r})$$

Obviously, f is analytic. Let C be the circle having center 0 and radius 1. Let γ be the curve which traverses C once ccw:

$$\gamma(\theta) = exp(i\theta), \qquad (0 \leq \theta \leq 2\pi)$$

Apply the Cauchy Integral Formula, for circular curves, to show that:

$$\frac{1}{1-r^2} = f_r(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-2rcos(\theta)+r^2}d\theta$$