MATHEMATICS 311
ASSIGNMENT 2
Due: February 11, 2015

01° Let Ω be a region in $\mathbf{C}$ and let f be a function defined on Ω with values in $\mathbf{C}$. Let u and v be the real and imaginary parts of f:

$$f = u + iv$$

Of course, the values of u and v lie in $\mathbf{R}$. In what follows, let us assume that the various partial derivatives of u and v exist, as required. Naturally:

$$\frac{\partial}{\partial x}f = \frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v, \qquad \frac{\partial}{\partial y}f = \frac{\partial}{\partial y}u + i\frac{\partial}{\partial y}v$$

One defines:

$$\frac{\partial}{\partial z}f = \frac{1}{2}(\frac{\partial}{\partial x}f - i\frac{\partial}{\partial y}f), \qquad \frac{\partial}{\partial \bar{z}}f = \frac{1}{2}(\frac{\partial}{\partial x}f + i\frac{\partial}{\partial y}f)$$

Verify that f is analytic iff:

$$\frac{\partial}{\partial \bar{z}}f = 0$$

in which case:

$$f' = \frac{\partial}{\partial z}f = \frac{\partial}{\partial x}f$$

One defines:

$$\triangle f = \frac{\partial^2}{\partial z \partial \bar{z}}f$$

Verify that:

$$\triangle f = \frac{1}{4}(\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f) \quad \text{and} \quad \triangle f = \frac{\partial^2}{\partial \bar{z} \partial z}f$$

One says that f is *harmonic* iff:

$$\triangle f = 0$$

Show that if f is analytic then f is harmonic. Show that if f is harmonic then:

$$\frac{\partial}{\partial z}f$$

is analytic.

02° Let $\mathbf{H}$ be the region in $\mathbf{C}$ consisting of all complex numbers z for which:

$$0 < Im(z)$$

One refers to $\mathbf{H}$ as the *upper half plane*. Let $\boldsymbol{\Delta}$ be the region in $\mathbf{C}$ consisting of all complex numbers z for which:

$$|z| < 1$$

One refers to $\boldsymbol{\Delta}$ as the *unit disk*. Let f be the Linear Fractional Transformation defined as follows:

$$f(z) = \frac{1}{i}\frac{z+i}{z-i}$$

Verify that $f(-i) = 0$, $f(0) = i$, $f(i) = \infty$, $f(-1) = -1$, and $f(1) = 1$. Show that f carries $\boldsymbol{\Delta}$ bijectively to $\mathbf{H}$. Describe the action of f on the boundary of $\boldsymbol{\Delta}$. Describe the inverse of f.

03° Let f be the function defined on $\mathbf{C}$ as follows:

$$f(z) = exp(-z^2) \qquad (z \in \mathbf{C})$$

Show that there is a function g defined and analytic on $\mathbf{C}$ such that:

$$g'(z) = f(z) \qquad (z \in \mathbf{C})$$

Conclude that, for any closed path γ in $\mathbf{C}$:

$$\int_\gamma f(z)dz = 0$$

04° Evaluate the Fresnel Integrals:

$$\lim_{r\to\infty} \int_0^r cos(u^2)du, \qquad \lim_{r\to\infty} \int_0^r sin(u^2)du$$

To do so, introduce the region Ω_r in $\mathbf{C}$ consisting of the complex numbers z for which:

$$|z| < r, \ \ 0 < arg(z) < \frac{\pi}{4}$$

In turn, introduce the simple closed path γ_r which traces the boundary of Ω_r counterclockwise. Finally, study the relation:

$$\int_{\gamma_r} exp(-z^2)dz = 0$$