MATHEMATICS 311
ASSIGNMENT 1
Due: February 4, 2015

01° For each z in $\mathbf{C}$, let M_z be the two by two matrix with real entries, defined as follows:

$$M_z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

where $z = x + iy$. Note that:

$$M_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that, for any z' and z'' in $\mathbf{C}$:

$$M_{z'+z''} = M_{z'} + M_{z''}, \quad M_{z'z''} = M_{z'}M_{z''}$$

From the polar form for z:

$$z = re^{i\theta} = rexp(i\theta) = r(cos(\theta) + isin(\theta))$$

show that:

$$M_z = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

Consequently, M_z defines a similarity transformation on $\mathbf{R}^2$, specifically, the composition of a rotation and a transformation of scale.

02• Let x be a real number for which:

$$sin(\frac{1}{2}x) \neq 0$$

Show that:

$$|\sum_{k=0}^{n-1} exp(ikx)|^2 = \frac{sin^2(\frac{1}{2}nx)}{sin^2(\frac{1}{2}x)}$$

03° For the following complex mapping:

$$w = f(z) = z^3 \qquad (z \in \mathbf{C})$$

verify the Cauchy/Riemann equations.

04° For the following complex mapping:

$$w = f(z) = \frac{z - i}{z + i} \qquad (z \in \mathbf{C},\ z \neq -i)$$

verify the Cauchy/Riemann equations.

05° For the following complex mapping:

$$w = f(z) = \frac{1}{z} \qquad (z \in \mathbf{C},\ z \neq 0)$$

verify the Cauchy/Riemann equations. Draw a diagram to show the relation between z and w.

06° For the following complex mapping:

$$w = exp(z) \qquad (z \in \mathbf{C})$$

verify the Cauchy/Riemann equations. In this context, we intend that:

$$w = u + iv, \quad z = x + iy$$

and:

$$u = e^x cos(y), \quad v = e^x sin(y)$$

07° For the following complex mapping:

$$w = log(z) \qquad (z \in \mathbf{C},\ 0 < |z|,\ Im(z) = 0 \Longrightarrow 0 < Re(z))$$

verify the Cauchy/Riemann equations. In this context, we intend that:

$$w = u + iv, \quad z = x + iy$$

and:

$$u = log(|z|), \quad v = arg(z)$$

See article 10• on page 3 of this assignment.

08° Find the real and imaginary parts of the complex number:

$$(1 + i)^{1+i} = exp((1 + i)log(1 + i))$$

09° Show that the complex mapping:

$$f(z) = |z| \qquad (z \in \mathbf{C},\ |z| \neq 0)$$

is not analytic.

10• Testify that you understand the following derivation of an expression for $arg(z)$. Let Ω be the principle domain for the logarithm function:

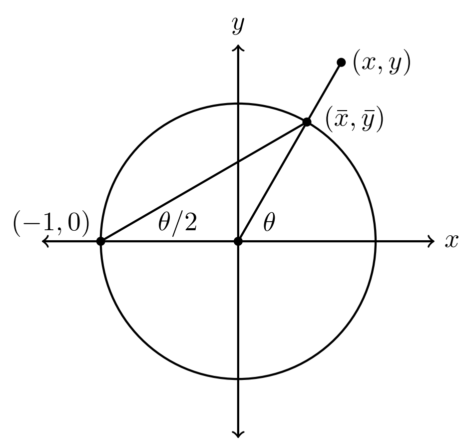
$$z \in \Omega \Longleftrightarrow (y = 0 \rightarrow 0 < x)$$

By definition:

$$log(z) = log(|z|) + i\, arg(z) \qquad (z \in \Omega)$$

By studying the following diagram, verify that:

$$(*) \qquad arg(z) = 2arctan(\frac{y}{x + |z|}) \qquad (z \in \Omega)$$



where of course:

$$z = x + i\, y, \quad \theta = arg(z)$$