MATHEMATICS 212
ASSIGNMENT 2
Due: February 11, 2015

01° Let α be a function defined on the interval $\mathbf{R}^+ := (0, \infty)$ in $\mathbf{R}$ and let ϕ be the scalar field defined on the region $D := \mathbf{R}^3 \backslash \{(0,0,0)\}$ in $\mathbf{R}^3$, as follows:

$$\phi(x,y,z) = \alpha(r) \qquad (0 < r := \sqrt{x^2+y^2+z^2})$$

Show that:

$$(\nabla\phi)(x,y,z) = \alpha^\circ(r)\frac{1}{r}(x,y,z) \qquad (0 < r := \sqrt{x^2+y^2+z^2})$$

02° Let β be a function defined on the interval $\mathbf{R}^+ := (0, \infty)$ in $\mathbf{R}$ and let F be the vector field defined on the region $D := \mathbf{R}^3 \backslash \{(0,0,0)\}$ in $\mathbf{R}^3$, as follows:

$$F(x,y,z) = \beta(r)(x,y,z) \qquad (0 < r := \sqrt{x^2+y^2+z^2})$$

Show that:

$$(\nabla \times F)(x,y,z) = (0,0,0) \qquad (0 < r := \sqrt{x^2+y^2+z^2})$$

Find a scalar field ϕ for which:

$$F(x,y,z) = (\nabla\phi)(x,y,z) \qquad (0 < r := \sqrt{x^2+y^2+z^2})$$

To do so, apply the foregoing problem. Work out the details for the following case:

$$\beta(r) = r^a \qquad (0 < r)$$

where a is any real number.

03° Let J be an interval in $\mathbf{R}$ and let D be a region in $\mathbf{R}^3$. Let ϕ be a scalar field "defined on D but depending on t":

$$\phi(t,x,y,z) \qquad (t \in J,\ \ (x,y,z) \in D)$$

One defines the following operator acting on ϕ:

(•) d'Alembertian

$$\Box^2\phi = \partial^2\phi/\partial t^2 - \nabla^2\phi = \partial^2\phi/\partial t^2 - \partial^2\phi/\partial x^2 - \partial^2\phi/\partial y^2 - \partial^2\phi/\partial z^2$$

Let G and H be vector fields "defined on D but depending on t":

$$G(t,x,y,z), \quad H(t,x,y,z) \qquad (t \in J,\ \ (x,y,z) \in D)$$

and satisfying the following relations:

$$\nabla \bullet G = 0, \quad \nabla \bullet H = 0$$

$$\partial G/\partial t - \nabla \times H = (0,0,0), \qquad \partial H/\partial t + \nabla \times G = (0,0,0)$$

Show that:

$$\square^2 G = (0,0,0) \quad \text{and} \quad \square^2 H = (0,0,0)$$

Of course, $\partial/\partial t$ and $\square^2$ act on G and H component by component. Conclude that any one of the components of G and H, let it be ϕ, satisfies the **wave equation**:

$$\square^2 \phi = 0$$

04• Let B be the (unit) rectangle in $\mathbf{R}^2$ comprised of all ordered pairs (x, y) for which:

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Let f be the (bounded) function defined on $\mathbf{R}^2$ as follows:

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin B \\ xy^2 & \text{if } (x,y) \in B \end{cases}$$

Apply the basic definition to show that f is integrable and that:

$$\iint f(x,y)dxdy = \frac{1}{6}$$