VECTORS

Representation of Vectors

1° Let X be any vector. With respect to a given Cartesian coordinate system, we may identify the initial point of X with the *origin* O of the system:

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The coordinates of the terminal point of X then determine X as an ordered triple of numbers:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Operations on Vectors

2° We may form the *sum* of vectors X and Y:

$$X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

and the *scalar product* of a number c and a vector Z:

$$c.Z = \begin{pmatrix} cz_1 \\ cz_2 \\ cz_3 \end{pmatrix}$$

We may form the *inner product* of vectors X and Y:

$$X \bullet Y = x_1y_1 + x_2y_2 + x_3y_3$$

and the *outer product*:

$$X \times Y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

as well. Finally, we may form the *length* of a vector X:

$$\|X\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Properties of the Operations

3° The foregoing operations have the following properties, among many others.

$$\begin{aligned}
(X+Y)+Z &= X+(Y+Z)\\
(a+b).Z &= a.Z+b.Z\\
(ab).Z &= a.(b.Z)
\end{aligned}$$

$$\begin{aligned}
X\bullet Y &= Y\bullet X\\
X\bullet (Y+Z) &= X\bullet Y+X\bullet Z\\
(a.X){\bullet}Y &= a(X\bullet Y)
\end{aligned}$$

$$\begin{aligned}
X\times Y &= -Y\times X\\
X\times (Y+Z) &= X\times Y+X\times Z\\
(a.X)\times Y &= a.(X\times Y)
\end{aligned}$$

$$\begin{aligned}
X\bullet (X\times Y) &= 0\\
&= (X\times Y)\bullet Y\\
X\bullet (Y\times Z) &= Y\bullet (Z\times X)\\
&= Z\bullet (X\times Y)\\
X\times (Y\times Z) &= (X\bullet Z).Y-(X\bullet Y).Z
\end{aligned}$$

$$\begin{aligned}
\|X\|^2 &= X\bullet X\\
X\bullet Y &= \|X\|\|Y\|cos(\theta)\\
\|X\times Y\| &= \|X\|\|Y\|sin(\theta)
\end{aligned}$$

In the last two relations, θ is the (unordered) angle between X and Y.

4° For any vector X:

$$X = x_1.E_1+x_2.E_2+x_3.E_3$$

where:

$$E_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad E_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad E_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$$