TAYLOR'S THEOREM
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1° Let n be a positive integer. Let A be an open subset of $\mathbf{R}^n$ and let f be a real-valued function defined on A, sufficiently smooth so that the following discussion makes sense. Let a be a member of A:

$$a = (a_1, a_2, \ldots, a_n)$$

Let k be a nonnegative integer. One defines the k-th order Taylor polynomial for f at a as follows:

$$(T_a^k f)(x) := \sum_{j=0}^{k} \frac{1}{j!}(H_a^j f)(x) \qquad (x \in \mathbf{R}^n)$$

where:

$$(H_a^j f)(x) := \sum_{|\gamma|=j} \frac{j!}{\gamma!}(D^\gamma f)(a)(x-a)^\gamma$$

In the foregoing expressions, we have applied the following notation. For each γ in $\mathbf{N}^n$:

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$$

one defines:

$$|\gamma| := \sum_{m=1}^{n} \gamma_m, \quad \gamma! := \prod_{m=1}^{n} \gamma_m!$$

For each γ in $\mathbf{N}^n$ and for each y in $\mathbf{R}^n$:

$$y = (y_1, y_2, \ldots, y_n)$$

one defines:

$$y^\gamma := \prod_{m=1}^{n} y_m^{\gamma_m}$$

Finally, for each γ in $\mathbf{N}^n$, one defines:

$$D^\gamma := \prod_{m=1}^{n} D_m^{\gamma_m}$$

where D_1, D_2, …, and D_n are the usual operators of differentiation:

$$D_m f := \frac{\partial}{\partial x_m} f \qquad (1 \leq m \leq n)$$

2° One defines the k-th order remainder function for f at a as follows:

$$(R_a^k f)(x) := f(x) - (T_a^k f)(x) \qquad (x \in A)$$

so that:

$$f(x) = (T_a^k f)(x) + (R_a^k f)(x) \qquad (x \in A)$$

3° Now let x be any member of A and let L be the line segment joining a and x. Let us assume that L is included in A, which would surely be true if x is "sufficiently near" a. Taylor's Theorem states that there must be some y in L such that:

$$(R_a^k f)(x) = \frac{1}{(k+1)!} \sum_{|\gamma|=k+1} \frac{(k+1)!}{\gamma!}(D^\gamma f)(y)(x-a)^\gamma$$

4° Let h be the function defined as follows:

$$h(t) = f((1-t)a + tx) \qquad (0 \leq t \leq 1)$$

Applying the elementary form of Taylor's Theorem to h, we find that there must be some u in $(0, 1)$ such that:

$$h(1) = \sum_{j=0}^{k} \frac{1}{j!} h^{(j)}(0)(1-0)^j + \frac{1}{(k+1)!} h^{(k+1)}(u)(1-0)^{k+1}$$

To prove the general form of Taylor's Theorem, we need only apply the Chain Rule to show that:

$$h^{(j)}(t) = \sum_{|\gamma|=j} \frac{j!}{\gamma!}(D^\gamma f)((1-t)a + tx)(x-a)^\gamma$$