MATHEMATICS 211
ASSIGNMENT 10
Due: November 19, 2014

01° Review the description of the Sinusoidal Map T in the previous assignment. Calculate the First Fundamental Form G for T:

$$G = \begin{pmatrix} T_u \bullet T_u & T_u \bullet T_v \\ T_v \bullet T_u & T_v \bullet T_v \end{pmatrix}$$

Show that:

$$det\, G = 1$$

Eventually, we will find that the foregoing condition implies that T preserves equal areas.

[Let A be the coordinate transformation which links the Hipparchus Map H and the Sinusoidal Map T:

$$A(\phi, \theta) = (u, v) = (\phi\, cos(\theta),\ \theta)$$

Obviously:

$$DA(\phi, \theta) = \begin{pmatrix} cos(\theta) & -\phi\, sin(\theta) \\ 0 & 1 \end{pmatrix}$$

We know that the first fundamental form F for H stands as follows:

$$F(\phi, \theta) = \begin{pmatrix} cos^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$

By class developments, we have:

$$det(F(\phi, \theta)) = det(DA(\phi, \theta)^t) det(G(u, v)) det(DA(\phi, \theta))$$

Consequently:

$$det(G(u, v)) = 1$$

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02° Calculate the curvature of the unit sphere $\mathbf{S}^2$ using the stereographic coordinate map S:

$$S(u, v) = (x, y, z) = (\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}) \quad ((u, v) \in \mathbf{R}^2)$$

[Apply the notebook: *Curvature.nb*. You should find the the curvature is constantly 1.]

03° Calculate the curvature of the northern hemisphere of the unit sphere $\mathbf{S}^2$ using the following coordinate map E:

$$E(u,v) = (x,y,z) = (u,v,\sqrt{1-u^2-v^2}) \quad (u^2+v^2<1)$$

[Apply the notebook: *Curvature.nb*. You should find the the curvature is constantly 1.]

04° Let J be any open interval in $\mathbf{R}$. Let f and g be real-valued functions defined on J for which:

$$0 < f(t), \quad \text{and} \quad f'(t)^2 + g'(t)^2 = 1$$

where t is any number in J. Note that:

$$f'(t)f''(t) + g'(t)g''(t) = 0$$

Let K be the open interval $(-\pi,\pi)$ in $\mathbf{R}$. Let H be the mapping carrying $J \times K$ to $\mathbf{R}^3$, defined as follows:

$$H(u,v) = (x,y,z) = (f(u)cos(v), f(u)sin(v), g(u))$$

where (u,v) is any ordered pair in $J \times K$. Let S be the surface in $\mathbf{R}^3$ parametrized by H:

$$S = H(J \times K)$$

Show that, for any ordered pair (u,v) in $J \times K$, the curvature $\kappa(u,v)$ of S at $H(u,v)$ has the form:

$$\kappa(u,v) = -\frac{f''(u)}{f(u)}$$

Now let $J = \mathbf{R}^+$. Design f and g so that, for any ordered pair (u,v) in $\mathbf{R}^+ \times \mathbf{R}^+$:

$$\kappa(u,v) = -1$$

To that end, introduce:

$$f(t) = exp(-t)$$

where t is any positive number. Then find a suitable function g. Sketch the graph of the corresponding surface S.

[Apply the notebook: *Curvature.nb*. You will need to apply the stated conditions on f and g, as indicated in the notebook. Given:

$$f(t) = exp(-t)$$

we obtain g as follows:

$$g'(t) = \sqrt{1-exp(-2t)}, \quad g(t) = \int_1^t \sqrt{1-exp(-2u)} + c$$

where c is any number.]