MATHEMATICS 211
ASSIGNMENT 9
Due: November 12, 2014

01° Let S be the Stereographic Coordinate Mapping for the Sphere $\mathbf{S}^2$, introduced by Ptolemy (cCE200):

$$S(u,v) = (x,y,z) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$$

where (u,v) is any ordered pair in $\mathbf{R}^2$. Calculate the Total Derivative for S:

$$DS(u,v) = \begin{pmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \\ z_u(u,v) & z_v(u,v) \end{pmatrix} = (\, P(u,v) \quad Q(u,v)\,)$$

and the First Fundamental Form for S:

$$G(u,v) = \begin{pmatrix} \alpha(u,v) & \beta(u,v) \\ \beta(u,v) & \gamma(u,v) \end{pmatrix} = \begin{pmatrix} P(u,v) \bullet P(u,v) & P(u,v) \bullet Q(u,v) \\ Q(u,v) \bullet P(u,v) & Q(u,v) \bullet Q(u,v) \end{pmatrix}$$

Note that:

$$G(u,v) = DS(u,v)^t DS(u,v)$$

Evaluate:

$$G(0,0)$$

[Apply the Mathematica notebook: Curvature.nb.]

02° Let a, b, and c be numbers in $\mathbf{R}^+$. Let $\mathbf{E}$ be the ellipsoidal surface in $\mathbf{R}^3$ parametrized by the Ellipsoidal Coordinate Map:

$$E(\phi,\theta) = (x,y,z) = (a\,cos(\theta)cos(\phi),\, b\,cos(\theta)sin(\phi),\, c\,sin(\theta))$$

where ϕ and θ are any numbers for which $-\pi < \phi < \pi$ and $-\pi/2 < \theta < \pi/2$, respectively. Calculate the curvature:

$$\kappa(\phi,\theta)$$

[Apply the Mathematica notebook: Curvature.nb.]

03° For the Sphere $\mathbf{S}^2$, let us recover the the Hipparchus Coordinate Map:

$$H : (\phi,\theta) \longrightarrow (x,y,z) = (cos\theta cos\phi, cos\theta sin\phi, sin\theta)$$

where:

$$(-\pi < \phi < \pi, -\frac{\pi}{2} < \theta < \frac{\pi}{2})$$

Find the coordinate transformations:

$$A:\ (\phi,\theta)\longrightarrow(u,v),\qquad B:\ (u,v)\longrightarrow(\phi,\theta)$$

which relate the Hipparchus and the Stereographic maps. We mean to say that:

$$H(\phi,\theta)=S(A(\phi,\theta))\quad\text{and}\quad S(u,v)=H(B(u,v))$$

Take care to describe the domains and ranges of these maps precisely.

[Bearing in mind that $(x,y,z)\neq(0,0,1)$ and that $x^2+y^2+z^2=1$, we find the stereographic coordinates (u,v) as follows:

$$(1-s)(0,0,1)+s(u,v,0)=(x,y,z),\ \ (1-t)(0,0,1)+t(x,y,z)=(u,v,0)$$

where s and t must be evaluated:

$$x=su,\ y=sv,z=1-s;s^2(u^2+v^2)+(1-s)^2=1$$
$$\Longrightarrow x=\frac{2u}{u^2+v^2+1},\ y=\frac{2v}{u^2+v^2+1},\ z=\frac{u^2+v^2-1}{u^2+v^2+1}$$

$$u=tx,\ v=ty,\ (1-t)+tz=0\implies u=\frac{x}{1-z},\ v=\frac{y}{1-z}$$

Now we have:

$$A(\phi,\theta)=(u,v)=(\frac{cos(\theta)cos(\phi)}{1-sin(\theta)},\frac{cos(\theta)sin(\phi)}{1-sin(\theta)})$$

Hence:

$$\frac{v}{u}=tan(\phi),\ \ \frac{u^2+v^2-1}{u^2+v^2+1}=z=sin(\theta)\quad(0<u)$$

Finally:

$$B(u,v)=(2arctan(\frac{v}{u+r}),arcsin(\frac{u^2+v^2-1}{u^2+v^2+1}))\quad(r=\sqrt{u^2+v^2})$$

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04° In practice, we regard the Hipparchus Coordinate Map as fundamental, we introduce, by imagination, a coordinate transformation:

$$A:\ (\phi,\theta)\longrightarrow(u,v)$$

and we proceed to define a new coordinate map T as follows:

$$T(u,v)=H(B(u,v))$$

For the coordinate transformation:

$$A(\phi, \theta) = (u, v) = (\phi cos\theta, \theta))$$

calculate the corresponding map T, the Sinusoidal Coordinate Map, a special case of the mapping just described. Of course, you must first calculate the (inverse) coordinate transformation B corresponding to A.

[We have:

$$B(u, v) = (\frac{u}{cos(v)}, v)$$

Hence:

$$T(u, v) = H(B(u, v)) = H(\frac{u}{cos(v)}, v)$$

Let FFF_H and FFF_T be the first fundamental forms for H and T, respectively. By computations in class, we know that:

$$det(FFF_H) = det(DA^t)det(FFF_T)det(DA)$$

so:

$$cos^2(\theta) = cos(\theta)det(FFF_T)cos(\theta)$$

Consequently:

$$det(FFF_T) = 1$$

See the next problem set.]

05• In the lectures, we will sketch a diagram which organizes visually the foregoing relations.