MATHEMATICS 211
ASSIGNMENT 8
Due: November 5, 2014

01° Let ρ, α, and β be positive numbers for which:

$$\alpha\sqrt{\rho^2+\beta^2}=1$$

Let Γ be the curve in $\mathbf{R}^3$ defined as follows:

$$\Gamma(s)=(\rho cos(\alpha s),\rho sin(\alpha s),\alpha\beta s)$$

where s is any number. Verify that:

$$\|\Gamma'(s)\|=1$$

where s is any number. Hence, s is the *arc-length* parameter. Let T, N, and B be the tangent, normal, and binormal vectors for Γ, respectively, defined as follows:

$$\begin{aligned} T(s)&=\Gamma'(s)\\ N(s)&=\frac{1}{\|T'(s)\|}T'(s)\\ B(s)&=T(s)\times N(s)\end{aligned}$$

where s is any number. By the Serret/Frenet formulae, we have:

$$\begin{aligned} T'(s)&=\quad\kappa(s)N(s)\\ B'(s)&=-\tau(s)N(s)\end{aligned}$$

where s is any number and where $\kappa(s)=\|T'(s)\|$ and $\tau(s)$ are the curvature and torsion, respectively, for Γ at $\Gamma(s)$. Calculate $\kappa(s)$ and $\tau(s)$. By explicit calculation, verify that:

$$(*)\qquad N'(s)=-\kappa(s)T(s)+\tau(s)B(s)$$

where s is any number.

[We find that:

$$T(s)=\Gamma'(s)=(-\alpha\rho\, sin(\alpha\, s),\alpha\rho\, cos(\alpha\, s),\alpha\beta)$$

Hence:

$$\|T(s)\|=\|\Gamma'(s)\|=\sqrt{\alpha^2\rho^2+\alpha^2\beta^2}=1$$

Moreover:

$$T'(s)=(-\alpha^2\rho\, cos(\alpha\, s),-\alpha^2\rho\, sin(\alpha\, s),0)$$

We obtain the curvature κ:

$$\kappa(s) = \|T'(s)\| = \alpha^2\rho$$

Hence:

$$N(s) = \frac{1}{\kappa(s)}T'(s) = (-cos(\alpha\, s), -sin(\alpha\, s), 0)$$

Now:

$$B(s) = T(s) \times N(s) = (\alpha\beta\, sin(\alpha\, s), -\alpha\beta\, cos(\alpha\, s), \alpha\rho)$$

so that:

$$B'(s) = (\alpha^2\beta\, cos(\alpha\, s), \alpha^2\beta\, sin(\alpha\, s), 0) = -\tau(s)N(s)$$

where τ is the torsion:

$$\tau(s) = \alpha^2\beta$$

Finally, we find that:

$$N'(s) = (\alpha\, sin(\alpha\, s), -\alpha\, cos(\alpha\, s), 0)$$

and:

$$\begin{aligned} & -\kappa(s)T(s) + \tau(s)B(s) \\ & = \\ & -\alpha^2\rho(-\alpha\rho\, sin(\alpha\, s), \alpha\rho\, cos(\alpha\, s), \alpha\beta) + \alpha^2\beta(\alpha\beta\, sin(\alpha\, s), -\alpha\beta\, cos(\alpha\, s), \alpha\rho) \end{aligned}$$

so relation $(*)$ holds true.]

02° Let Γ be the curve in $\mathbf{R}^3$ defined as follows:

$$\Gamma(s) = (\frac{\sqrt{1+s^2}}{\sqrt{5}}, \frac{2s}{\sqrt{5}}, \frac{log(s+\sqrt{1+s^2})}{\sqrt{5}})$$

where s is any number. Repeat the steps in the foregoing problem.

[See the Mathematica notebook *SpaceCurves.nb.*]

03° For a curve parametrized by arclength, the Serret/Frenet formulae stand as follows:

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

Let:

$$A = \tau T + \kappa B$$

Show that:

$$\begin{aligned} T' &= A \times T \\ N' &= A \times N \\ B' &= A \times B \end{aligned}$$

[The relevant relations are:

$$T \times N = B, \quad N \times B = T, \quad B \times T = N$$

These relations hold for any orthonormal triad. Now we obtain:

$$A \times T = (\tau T + \kappa B) \times T = \tau T \times T + \kappa B \times T = 0 + \kappa N = T'$$

and so forth.]

04° Let a be a positive number. The Curve of Viviani traces (part of) the intersection of the cylinder:

$$(x - a)^2 + y^2 = a^2$$

and the sphere:

$$x^2 + y^2 + z^2 = (2a)^2$$

in $\mathbf{R}^3$. One may parametrize the curve as follows:

$$\Gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = a \begin{pmatrix} 1 + cos(t) \\ sin(t) \\ 2sin(t/2) \end{pmatrix} \qquad (0 \leq t \leq \pi)$$

Note that the parameter t is not the arclength parameter. Find the curvature κ and the torsion τ of the Curve of Viviani. To do so, you may (if you wish) apply the following general formulas:

$$\kappa(t) = \frac{1}{\|\Gamma'(t)\|^3} \|\Gamma'(t) \times \Gamma''(t)\|$$

$$\tau(t) = \frac{1}{\|\Gamma'(t) \times \Gamma''(t)\|^2} (\Gamma'(t) \times \Gamma''(t)) \bullet \Gamma'''(t)$$

[See the Mathematica notebook *SpaceCurves.nb.*]