MATHEMATICS 211
ASSIGNMENT 7
Due: October 29, 2014

01° Let f be the function defined as follows:

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + (x - \frac{3}{2})^2 - (y+4)^2$$

where (x,y) is any point in $\mathbf{R}^2$. Find the critical points for f. That is, find the points (a,b) for which:

$$f_x(a,b) = 0, \quad f_y(a,b) = 0$$

For each such point (a,b), determine whether it is a local minimum point, a saddle point, or a local maximum point. Of course, a priori, it might be none of the three.

[We have:

$$f_x(x,y) = x^2 + 2x - 3, \;\; f_y(x,y) = y^2 - 2y - 8$$

and:

$$f_{xx}(x,y) = 2x+2, \;\; f_{xy}(x,y) = 0 = f_{yx}(x,y) = 0, \;\; f_{yy}(x,y) = 2y - 2$$

By applying the Quadratic Equation, we find the critical points:

$$(-3,-2),\; (-3,4),\; (1,-2),\; (1,4)$$

The corresponding Hessians are:

$$\begin{pmatrix} -4 & 0 \\ 0 & -6 \end{pmatrix}, \begin{pmatrix} -4 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

The first point is a local maximum and the last a local minimum, while the second and third are saddle points, because the determinant is negative.]

02° Let f be the function defined as follows:

$$f(x,y) = xy(4x^2 + y^2 - 16)$$

where (x,y) is any point in $\mathbf{R}^2$ for which:

$$0 \leq x, \quad 0 \leq y, \quad 4x^2 + y^2 \leq 16$$

Find the global minimum and maximum values for f.

[Clearly, the maximum value is 0. It occurs at every point on the periphery of the domain. At interior points, the values are negative. By the Extreme Value Theorem, there must be at least one point in the interior at which the minimum value occurs. At such a point, let it be (a, b), we have:

$$0 = f_x(a, b) = b(4a^2 + b^2 - 16) + 8a^2b$$
$$0 = f_y(a, b) = a(4a^2 + b^2 - 16) + 2ab^2$$

Obviously

$$8a^2 = 2b^2, \quad b = 2a, \quad a = 1, \quad (a, b) = (1, 2), \quad f(1, 2) = -16$$

]

03° Let f be the function defined as follows:

$$f(x, y) = 6xy^2 - 2x^3 - 3y^4$$

where (x, y) is any point in $\mathbf{R}^2$. Find the three critical points for f. For each such point (a, b), determine whether it is a local minimum point, a saddle point, or a local maximum point. For one of the points, you will need to exercise ingenuity.

[This problem is very like the first, but the critical point $(0, 0)$ presents a problem. The corresponding Hessian is:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So the tests fail. However, along the lines $(x, 0)$, $(x, -x)$, and (x, x), the values of f show an inflection, while along the line $(0, y)$, a (local) maximum.]

04° Show that, among all triangles inscribed in a given circle, the equilateral triangles have the greatest perimeter.

[The problem involves a constrained extremum:

$$f(\alpha, \beta, \gamma) = 2sin(\frac{\alpha}{2}) + 2sin(\frac{\beta}{2}) + 2sin(\frac{\gamma}{2}), \quad g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma = \pi$$

We apply Lagrange's Method:

$$(\nabla f)(\alpha, \beta, \gamma) = \lambda(\nabla g)(\alpha, \beta, \gamma)$$

which yields:

$$(cos(\frac{\alpha}{2}), cos(\frac{\beta}{2}), cos(\frac{\gamma}{2})) = \lambda(1, 1, 1)$$

so:

$$\alpha = \beta = \gamma = \frac{\pi}{3} \quad]$$