MATHEMATICS 211
ASSIGNMENT 6
Due: October 15, 2014

01° Let N be the vector in $\mathbf{R}^3$ defined as follows:

$$N \equiv \frac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (\|N\| = 1)$$

Let Λ be the *reflection* on $\mathbf{R}^3$ defined by N:

$$\Lambda(X) \equiv X - 2(X \bullet N)N \qquad (X \in \mathbf{R}^3)$$

Note that Λ is linear. Find the matrix for Λ. Compute the determinant of the matrix.

[Let c stand for $(1/3)\sqrt{3}$. We find that:

$$\Lambda(E_1) = E_1 - 2cN,\ \Lambda(E_2) = E_2 - 2cN,\ \Lambda(E_3) = E_3 - 2cN$$

Consequently:

$$\Lambda = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

Hence:

$$det(\Lambda) = -1$$

]

02° Again, let N be the vector in $\mathbf{R}^2$ defined as follows:

$$N \equiv \frac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and let A be the corresponding antisymmetric matrix:

$$A \equiv \frac{\sqrt{3}}{3}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Compute the matrix A^2. Let θ be any real number. Let R be the ccw *rotation* about the axis $\mathbf{R}N$ through the angle θ, defined as follows:

$$R \equiv exp(\theta A) = I + sin(\theta)A + (1 - cos(\theta))A^2$$

(See problem $05^\bullet$, where the definition of R is "justified.") For the case in which $\theta = \pi/2$, compute the matrix R explicitly. Then, for the vector:

$$X \equiv \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

calculate:

$$Y = RX$$

[We have:

$$\begin{aligned} R &= I + A + A^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} + c^2 \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \end{aligned}$$

where $c = (1/3)\sqrt{3}$. Let $a = (1/3)(1 - \sqrt{3})$ and $b = (1/3)(1 + \sqrt{3})$. Then:

$$R = \begin{pmatrix} 1/3 & a & b \\ b & 1/3 & a \\ a & b & 1/3 \end{pmatrix}$$

We obtain (by Mathematica):

$$Y = RX = c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad Y \bullet X = 0 = cos(\frac{\pi}{2}) \quad]$$

03° Let M be any matrix having three rows and three columns:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

One defines the *trace* of M as follows:

$$tr(M) = m_{11} + m_{22} + m_{33}$$

Now let M' and M'' be any matrices having three rows and three columns. Show that:

$$tr(M'M'') = tr(M''M')$$

[Apply patient computation.]

04° Let M be a matrix with two rows and two columns:

$$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

Let f be the quadratic polynomial defined as follows:

$$f(x) = det(xI - M) = det\begin{pmatrix} x-p & r \\ q & x-s \end{pmatrix}$$

Apply the Quadratic Formula to factor f:

$$f(x) = (x-u)(x-v)$$

where u and v are the zeros of f. These zeros are called the *characteristic values* of M. Verify that:

$$tr(M) = u + v, \ \ det(M) = uv$$

[Obviously:

$$f(x) = x^2 - (p+s)x + (ps - qr) = (x-u)(x-v) = x^2 - (u+v)x + uv$$

where u and v are the zeros of f, determined by the Quadratic Equation of Olde. The conclusions follow.]

05• Let A be a matrix having 3 rows and 3 columns. One defines $exp(A)$ as follows:

$$exp(A) \equiv \sum_{j=0}^{\infty} \frac{1}{j!}A^j = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \frac{1}{24}A^4 + \ \cdots$$

In particular, let A be *antisymmetric*:

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

We find that, for any vectors X and Y in $\mathbf{R}^3$:

(1) $$AX \bullet Y = -X \bullet AY$$

Now let $a^2 + b^2 + c^2 = 1$. Obviously:

(2) $$A^3 = -A \quad \text{hence} \quad A^4 = -A^2$$

Now it is plain that, for each real number t:

$$(3) \qquad exp(tA) = I + sin(t)A + (1 - cos(t))A^2$$

We plan to show that $exp(tA)$ is the ccw rotation carrying $\mathbf{R}^3$ to itself, for which the axis of rotation is the line $\mathbf{R}N$ passing through the origin O and for which the angle of rotation is t. To that end, we note that:

$$(4) \qquad \frac{d}{dt} exp(tA) = A\, exp(tA)$$

By (1), (2), and (3) or by (4) alone, we find that, for any vectors X and Y in $\mathbf{R}^3$:

$$(5) \qquad exp(tA)X \bullet exp(tA)Y = X \bullet Y$$

Now we can say that $exp(tA)$ preserves inner products. It also preserves norms. That is, for any vector Z in $\mathbf{R}^3$:

$$\|exp(tA)Z\|^2 = exp(tA)Z \bullet exp(tA)Z = Z \bullet Z = \|Z\|^2$$

Let:

$$N \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We note that $AN = N \times N = 0$, hence that:

$$(6) \qquad exp(tA)N = N$$

Let X be any vector in $\mathbf{R}^3$ for which $X \bullet N = 0$ and let $Y \equiv exp(tA)X$. Of course, $\|Y\| = \|X\|$. Hence $X \bullet X = \|X\|\|Y\|$. By (5) and (6), $Y \bullet N = 0$. We note that, by (1), $X \bullet AX = 0$ and (by computation of A^2) that $(I + A^2)X = 0$. Hence, $A^2 X = -X$. Now we verify that:

$$(7) \qquad X \bullet Y = cos(t)\, X \bullet X = cos(t)\|X\|\,\|Y\|$$

which entails that the angle between X and Y is t. Finally, we conclude that:

$$exp(tA) = I + sin(t)A + (1 - cos(t))A^2$$

is the ccw rotation carrying $\mathbf{R}^3$ to itself, for which the axis of rotation is the line $\mathbf{R}N$ passing through the origin O and for which the angle of rotation is t.