MATHEMATICS 211
ASSIGNMENT 4
Due: October 1, 2014

01° Let L be the linear mapping carrying $\mathbf{R}^3$ to $\mathbf{R}^2$ for which the matrix relative to the standard bases:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively, stands as follows:

$$L = \begin{pmatrix} -1 & 12 & 10 \\ 6 & 6 & 18 \end{pmatrix}$$

Find the *nullspace* $\mathcal{N}(L)$ for L, composed of all vectors X:

$$X = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

in $\mathbf{R}^3$ for which:

$$L(X) = \begin{pmatrix} -1 & 12 & 10 \\ 6 & 6 & 18 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Show that, in fact, $\mathcal{N}(L)$ is a line in $\mathbf{R}^3$ passing through the origin. Find the *rangespace* $\mathcal{R}(L)$ for L, composed of all vectors Y:

$$Y = \begin{pmatrix} p \\ q \end{pmatrix}$$

in $\mathbf{R}^2$ for which there exists a vector X:

$$X = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

in $\mathbf{R}^3$ such that:

$$L(X) = \begin{pmatrix} -1 & 12 & 10 \\ 6 & 6 & 18 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = Y$$

Show that, in fact, $\mathcal{R}(L) = \mathbf{R}^2$.

[We find that if X lies in $\mathcal{N}(L)$ and if $w = 1$ then u and v must be -2 and -1, respectively. Hence, X must stand in the form:

$$X = t\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

where t is any number. Now we search for vectors A and B in $\mathbf{R}^3$ such that:

$$L(A) = E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad L(B) = E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Succeeding, we would find that, for any vector Y in $\mathbf{R}^2$:

$$L(pA + qB) = \begin{pmatrix} p \\ q \end{pmatrix} = Y$$

Consequently, $\mathbf{R}(L) = \mathbf{R}^2$. We may find such vectors A and B by straightforward elimination.]

02° Let L be the mapping carrying $\mathbf{R}^2$ to $\mathbf{R}^3$, defined as follows:

$$L(\begin{pmatrix} s \\ t \end{pmatrix}) = (s - t)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (s + t)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where s and t are any real numbers. Note that L a linear mapping. Find the matrix:

$$\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix}$$

which defines L.

[We produce the columns of the required matrix as follows:

$$L(E_1) = L(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad L(E_2) = L(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

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03° Calculate the determinant of the following matrix:

$$\begin{pmatrix} -1 & 3 & 2 & 1 \\ 2 & -3 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 4 & 1 & 1 & -1 \end{pmatrix}$$

To that end, apply the characteristic properties of determinants.

[Mathematica says -62.]

04° Calculate the determinant of the following *rook placement* matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

[We must interchange columns three times to obtain the identity matrix, so the determinate is -1]

05° Let L be the linear mapping carrying $\mathbf{R}^2$ to $\mathbf{R}^2$, defined by the following matrix, having 2 rows and 2 columns:

$$L = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

where a, b, c, and d are any real numbers. Let A be the subset of $\mathbf{R}^2$ consisting of all vectors:

$$X = \begin{pmatrix} u \\ v \end{pmatrix}$$

for which $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Let B be the image of A under L, consisting of all vectors:

$$Y = \begin{pmatrix} p \\ q \end{pmatrix}$$

in $\mathbf{R}^2$ for which there is some vector X:

$$X = \begin{pmatrix} u \\ v \end{pmatrix}$$

in A such that:

$$L(X) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = Y$$

Show that the area of B equals:

$$|ad - bc| = |det(L)|$$

[Of course, B is the parallelogram with vertices at the positions:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

in $\mathbf{R}^2$. Let us introduce the vectors:

$$P = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \ Q = \begin{pmatrix} c \\ d \\ 0 \end{pmatrix}, \quad \text{hence} \quad P \times Q = \begin{pmatrix} 0 \\ 0 \\ ad - bc \end{pmatrix}$$

in $\mathbf{R}^3$. Drawing a simple diagram, we find that the area of B equals:

$$\|P\|\|Q\|sin(\theta)$$

where θ is the angle between P and Q. (To that end, we need only "drop the perpendicular.") In turn, we have:

$$\begin{aligned} (ad - bc)^2 &= (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 \\ &= \|P\|^2\|Q\|^2 - \langle\!\langle P, Q \rangle\!\rangle^2 \\ &= \|P\|^2\|Q\|^2(1 - cos^2(\theta)) \\ &= \|P\|^2\|Q\|^2 sin^2(\theta) \end{aligned}$$

It follows that the area of B equals $|ad - bc|$.]

06° Let a, b, and c be any numbers. Show that:

$$det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (c-b)(c-a)(b-a)$$

[Applying the basic definition, we find that:

$$det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (bc^2 - cb^2) - (ac^2 - ca^2) + (ab^2 - ba^2) = (c-b)(c-a)(b-a)$$

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07$^\bullet$ Let c and d be positive constants. Let E be the subset of $\mathbf{R}^2$ composed of all positions:

$$Z = \begin{pmatrix} x \\ y \end{pmatrix}$$

in $\mathbf{R}^2$ such that:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = d$$

In terms of c and d, find the positive constants a and b such that, for any position:

$$Z = \begin{pmatrix} x \\ y \end{pmatrix}$$

in $\mathbf{R}^2$, Z lies in E iff:

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$$

You should express a and b in terms of c and d. One refers to E as an *ellipse* with *focii* at:

$$\begin{pmatrix} -c \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ 0 \end{pmatrix}$$

Draw a picture of E, displaying the focii and indicating the significance of a and b.

[This problem provides our first introduction to the *ellipse*.]