MATHEMATICS 211
ASSIGNMENT 3
Due: September 24, 2014

01° Let J be an open interval in $\mathbf{R}$. Let f, g, and h be differentiable functions defined on J with values in $\mathbf{R}$, for which:

$$(1) \qquad \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix} \times \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (t \in J)$$

Let Γ be the corresponding mapping carrying J to $\mathbf{R}^3$, with components f, g, and h:

$$\Gamma(t) \equiv \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix} \qquad (t \in J)$$

Let Γ satisfy the Equation of Newton:

$$(2) \qquad \Gamma''(t) \equiv \begin{pmatrix} f''(t) \\ g''(t) \\ h''(t) \end{pmatrix} = -\frac{1}{\|\Gamma(t)\|^3}\Gamma(t) \qquad (t \in J)$$

Show that the range of Γ is included in a plane. To that end, compute the derivative of:

$$\Delta(t) = \Gamma'(t) \times \Gamma(t)$$

[Clearly:

$$\Delta'(t) = \Gamma'(t) \times \Gamma'(t) + \Gamma''(t) \times \Gamma(t)$$

By our assumption (2), the Equation of Newton, $\Gamma''(t)$ is a scalar multiple of $\Gamma(t)$. Hence:

$$\Delta'(t) = 0$$

It follows that Δ is constant:

$$\Delta(t) = C$$

where C is a vector in $\mathbf{R}^3$. By our initial assumption (1), we know that $C \neq 0$. We know that;

$$\langle\!\langle\, \Gamma(t), C \,\rangle\!\rangle = 0$$

Hence, the position $\Gamma(t)$ must lie in the plane in $\mathbf{R}^3$ composed of all vectors Y for which:

$$\langle\!\langle\, C, Y \,\rangle\!\rangle = 0$$

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02° Let X and Y be nonempty closed subsets of $\mathbf{R}^2$ for which $X \cap Y = \emptyset$. Let d be the *distance* between X and Y, defined as follows:

$$d = \inf\{\|x - y\| \,:\, x \in X,\, y \in Y\}$$

Show by example that d may be 0 (even though X and Y have no point(s) in common). For contrast, show that if X or Y is compact then d is in fact positive.

[For the first part of the problem, we need only take X and Y to be the sets:

$$X = \{(x, y) : y = 0)\}, \quad Y = \{(x, y) : 0 < x,\, y = \frac{1}{x}\}$$

Clearly, $d = 0$. Now let X be not only closed but compact. We must show that $0 < d$. We argue by contradiction. Let us suppose that $d = 0$. Under this supposition, there would exist sequences:

$$\xi : \quad x_1, x_2, x_3, \ldots\, x_j. \,\ldots, \qquad \eta : \quad y_1, y_2, y_3, \ldots\, y_j. \,\ldots$$

in X and Y, respectively, such that:

$$d(x_j, y_j) \longrightarrow 0$$

Since X is compact, we may introduce a convergent subsequence of ξ and we may introduce the corresponding subsequence of η. For simplicity of exposition, we might just as well presume that ξ itself is convergent and we may take the corresponding subsequence of η to be η itself. Now the sequences ξ and η converge to a common limit, let it be w. Since X and Y are closed, w must lie in $X \cap Y$, contradicting our initial condition X and Y be disjoint. Consequently, our supposition that $d = 0$ must be untenable. We conclude that $0 < d$.]

03° Let P be the subset of $\mathbf{R}^2$ consisting of all positions:

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

for which $y = x^2$. Let τ be the position:

$$\tau = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

in $\mathbf{R}^2$. Find the distance between P and $\{\tau\}$.

[We apply the methods of Elementary Calculus. We present the (squares of the) distances between the positions in S and the position τ as follows:

$$f(t) = (t-2)^2 + (t^2+1)^2$$

where t is any real number. Clearly, we need consider only values of t for which:

$$0 \leq t$$

We find that:

$$f'(t) = 4t^3 + 6t - 4, \quad f''(t) = 12y^2 + 6$$

Obviously, f' is strictly increasing. Consequently, we may introduce a positive number u, uniquely determined by the condition:

$$f'(u) = 0 \quad \text{that is,} \quad u^3 + \frac{3}{2}u = 1$$

We conclude that the distance between P and $\{\tau\}$ is:

$$f(u) = (u-2)^2 + (u^2+1)^2$$

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04• Let $\mathbf{S}^2$ be the *unit sphere* in $\mathbf{R}^3$, consisting of all positions:

$$x = (x_1, x_2, x_3)$$

for which:

$$x_1^2 + x_2^2 + x_3^2 = 1$$

Let a, b, c, and d be any four positions:

$$\begin{aligned} a &= (a_1, a_2, a_3) \\ b &= (b_1, b_2, b_3) \\ c &= (c_1, c_2, c_3) \\ d &= (d_1, d_2, d_3) \end{aligned}$$

in $\mathbf{S}^2$ for which:

$$(*) \qquad a \neq b,\ a \neq c,\ a \neq d,\ b \neq c,\ b \neq d,\ c \neq d$$

Let f be the function of the foregoing four positions, defined as follows:

$$f(a,b,c,d) = \frac{1}{\|a-b\|} + \frac{1}{\|a-c\|} + \frac{1}{\|a-d\|} + \frac{1}{\|b-c\|} + \frac{1}{\|b-d\|} + \frac{1}{\|c-d\|}$$

Of course, the domain of f would be the subset $\mathbf{\Sigma}$ of:

$$\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 = \mathbf{R}^{12}$$

consisting of all quadruples:

$$(a, b, c, d)$$

of positions in $\mathbf{S}^2$ which satisfy condition $(*)$. Show that the range of f has the form:

$$[\ell, \longrightarrow)$$

where ℓ is a suitable positive (!) real number. We mean to say that ℓ is the minimum value of f but that the values of f are arbitrarily large. Guess the form of the various quadruples (a, b, c, d) in $\mathbf{\Sigma}$ for which:

$$f(a, b, c, d) = \ell$$

Start by guessing the "shape" of such quadruples.

05$^\bullet$ Reduce the foregoing problem to pairs and triple of positions in $\mathbf{S}^2$. Then generalize the foregoing problem to k-tuples of positions in $\mathbf{S}^2$, where k is any positive integer $(2 \leq k)$.