**MATHEMATICS 211**
ASSIGNMENT 2
Due: September 17, 2014

01° Let $\xi$:

$$\xi : \quad x_1, x_2, x_3, \ \ldots$$

be a sequence in $\mathbf{R}^2$ defined as follows:

$$x_j = (cos((2j-1)\frac{\pi}{4}), sin((2j-1)\frac{\pi}{4}))$$

where $j$ is any positive integer. Show that $\xi$ is not convergent. In turn, describe a subsequence $\eta$:

$$\eta : \quad y_1, y_2, y_3, \ \ldots$$

of $\xi$ which is in fact convergent. Of course, there are many.

[ The sequence $\xi$ follows a cycle of length 4:

$$\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(-1,1), \frac{1}{\sqrt{2}}(-1,-1), \frac{1}{\sqrt{2}}(1,-1)$$

Starting with any one of the four positions, one may produce a (constant) subsequence of $\xi$ by following every fourth term. As there are four such subsequences (and many others), it is plain that $\xi$ itself cannot be convergent. (If it were, then every subsequence of $\xi$ would necessarily converge to the same limit, namely, the limit of $\xi$.) ]

02° Let $S$ be the subset of $\mathbf{R}^2$ consisting of all positions:

$$x = (u, v)$$

such that:

$$0 < u^2 + v^2 \leq 1$$

Show that $S$ is neither open nor closed.

[ Obviously, there is a sequence $\xi$ in $S$ which converges to $(0,0)$, namely:

$$\xi : \quad x_j = (\frac{1}{j+1}, \frac{1}{j+1}) \qquad (j \in \mathbf{Z}^+)$$

Since $(0,0) \notin S$, we infer that $S$ is not closed. Moreover, $(1,0) \notin int(S)$, because, for each positive number $r$, $B_r((1,0)$ contains positions in the complement of $S$. We infer that $S$ is not open. ]

03° Let $T$ be a subset of $\mathbf{R}^2$ such that:

$$T \neq \emptyset, \quad \mathbf{R}^2 \backslash T \neq \emptyset$$

Show that the periphery of $T$ is not empty:

$$per(T) \neq \emptyset$$

[Let us denote $\mathbf{R}^2 \backslash T$ by $\bar{T}$. Let $x$ be a position in $T$ and let $y$ be a position in $\bar{T}$. Let $d = \|x - y\|$. Let $A$ be the subset of $[0, d]$ consisting of all numbers $a$ such that:

$$x + \frac{a}{d}(y - x) \in T$$

Obviously, $0 \in A$ and $d$ is an upper bound for $A$. Consequently, we may introduce the supremum (that is, the least upper bound) for $A$. Let it be $b$. Let:

$$v = x + \frac{b}{d}(y - x)$$

We will show that $v \in per(T)$. Let $r$ be any positive number. We must show that:

(1) $$B_r(v) \cap T \neq \emptyset$$

and:

(2) $$B_r(v) \cap \bar{T} \neq \emptyset$$

Clearly, $b - r$ cannot be an upper bound for $A$, since it is smaller than the least upper bound $b$. Hence, there must be some $c$ in $A$ such that $b - r < c \leq b$. Let:

$$u = x + \frac{c}{d}(y - x)$$

It follows that:

$$u \in T \quad \text{and} \quad \|u - v\| = |c - b| < r$$

We infer that (1) holds true. In turn, there must be some $c$ in $(b, d]$ (unless $b = d$) such that $c \notin A$ and $b \leq c < b + r$. Let:

$$w = x + \frac{c}{d}(y - x)$$

It follows that:

$$w \in \bar{T} \quad \text{and} \quad \|w - v\| = |c - b| < r$$

We infer that (2) holds true. For the outstanding case in which $b = d$, we simply note that $v \in \bar{T}$, so that, again, (2) holds true.

04• To support the foregoing problem, we supply the following discussion of *topology* on $\mathbf{R}^n$. Let $S$ be any subset of $\mathbf{R}^n$. Relative to $S$, we obtain the following partition of $\mathbf{R}^n$:

$$\mathbf{R}^n = int(S) \cup per(S) \cup ext(S)$$

We refer to $int(S)$, $per(S)$, and $ext(S)$ as the *interior*, the *periphery*, and the *exterior* of $S$, respectively. They are defined as follows:

$$\begin{aligned} int(S) &= \{x \in \mathbf{R}^n \ : \ (\exists r > 0)(B_r(x) \subseteq S\} \\ perS) &= \{x \in \mathbf{R}^n \ : \ (\forall r > 0)(B_r(x) \cap S \neq \emptyset \ \wedge \ B_r(x) \cap \mathbf{R}^n \backslash S \neq \emptyset\} \\ ext(S) &= \{x \in \mathbf{R}^n \ : \ (\exists r > 0)(B_r(x) \subseteq \mathbf{R}^n \backslash S\} \end{aligned}$$

In the foregoing context, we have applied the common notation $B_r(x)$ for the *open ball* with center $x$ and radius $r$:

$$B_r(x) = \{y \in \mathbf{R}^n \ : \ \|y - x\| < r\}$$

We define the *closure* $clo(S)$ of $S$ to be the union of the interior and the periphery:

$$clo(S) = int(S) \cup per(S)$$

Obviously:

$$int(S) \subseteq S \subseteq clo(S)$$

We say that $S$ is *open* iff $S = int(S)$ and that $S$ is *closed* iff $S = clo(S)$. At this point, one should test understanding by proving that $S$ is open iff $\mathbf{R}^n \backslash S$ is closed. We say that $S$ is *bounded* iff:

$$(\exists r > 0)(S \subseteq B_r(0))$$

Finally, we say that $S$ is *compact* iff $S$ is closed and bounded.

05• The term *topology* is a concatenation of the Greek words *topos* ($\tau o \pi o \sigma$) and *logos* ($\lambda o \gamma o \sigma$), the former referring to "position" and the latter in general to "word" but in particular to "explanation." The term evolved into the Latin form *analysis situs*.

06• Let $\xi$

$$\xi : \quad x_1, x_2, x_3, \ \ldots$$

be a sequence in $\mathbf{R} = \mathbf{R}^1$. Show that there must exist a subsequence $\eta$:

$$\eta : \quad y_1, y_2, y_3, \ \ldots$$

of $\xi$ such that $\eta$ is decreasing or $\eta$ is increasing. To that end, introduce the concept of a "leader." For each positive integer $j$, one says that $j$ is a "leader" for $\xi$ iff, for each positive integer $k$, if $j \leq k$ then $x_k \leq x_j$. Let $L$ be the subset of $\mathbf{Z}^+$ consisting of all leaders for $\xi$. Show that if $L$ is finite then there must be a subsequence $\eta$ of $\xi$ such that $\eta$ is increasing, while if $L$ is infinite then there must be a subsequence $\eta$ of $\xi$ such that $\eta$ is decreasing.