EXAMINATION

MATHEMATICS 211

Due: L306, HIGH NOON, FRIDAY, DECEMBER 19, 2014

01° Let T be a regular tetrahedron. Let A and B be two of the vertices of T and let O be its center. Find the angle between the vector X joining O to A and the vector Y joining O to B.

02° Let r, s, u, v, and w be real variables which meet the following conditions:

$$1 < r$$
$$s = (r-1)exp(r)$$
$$s = \frac{1}{4}(v^2 - u^2)$$
$$w = \frac{1}{r}exp(-r)$$

Show that:

$$\frac{\partial w}{\partial u} = \frac{1}{2}(1 + \frac{1}{r})uw^2$$
$$\frac{\partial w}{\partial v} = -\frac{1}{2}(1 + \frac{1}{r})vw^2$$

To do so, first show that:

$$\frac{dr}{ds} = w$$

03° Let f be the function defined on the open first octant in $\mathbf{R}^3$, as follows:

$$f(x, y, z) \equiv x^{1/2} + y^{1/2} + z^{1/2} \qquad (0 < x,\ 0 < y,\ 0 < z)$$

Let d be any positive real number and let S be the surface in $\mathbf{R}^3$ defined by the condition:

$$f(x, y, z) = d^{1/2}$$

Let (x, y, z) be any point in S and let Π be the tangent plane to S at (x, y, z). Let:

$$(p, 0, 0),\ \ (0, q, 0),\ \ (0, 0, r)$$

be the points on the coordinate axes which lie in Π. Show that:

$$p + q + r = d$$

04° Let a, b, and c be any positive real numbers. Let f be the function defined on the open first quadrant in $\mathbf{R}^2$, as follows:

$$f(x, y) \equiv \frac{a}{x} + bxy + \frac{c}{y} \qquad (0 < x,\ 0 < y)$$

Show that there is precisely one critical point for f. Show that the critical point is a local minimum. Is it a global minimum?

05° Let a, b, and c be positive numbers and let u, v, w, and d be any numbers for which $u^2 + v^2 + w^2 \neq 0$. Find the minimum distance between the ellipsoid E:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the plane P:

$$ux + vy + wz = d$$

Of course, the answer will depend on the given parameters. Be wary of degenerate cases.

06° Let C be the curve in $\mathbf{R}^3$, parametrized by the mapping Γ defined as follows:

$$\Gamma(t) \equiv (\,cosh(t), 0,\, sinh(t)\,) \qquad (t \in \mathbf{R})$$

Let S be the surface in $\mathbf{R}^3$, parametrized by the mapping H defined as follows:

$$H(u,v) \equiv (\,cosh(u)\,cos(v),\; cosh(u)\,sin(v),\; sinh(u)\,) \quad (u \in \mathbf{R},\; -\pi < v < \pi)$$

Draw a diagram to show that one may regard S as the *surface of revolution* defined by the *profile curve* C. Find the curvature $\kappa(u,v)$ of S at the position $H(u,v)$. Why is the curvature independent of v?

07° Let a, b, and c be any positive numbers. Let S be the subset of $\mathbf{R}^3$ consisting of all points (x,y,z) such that:

$$0 < x,\; 0 < y,\; 0 < z,\; x^a y^b z^c = 1$$

Let f be the function defined on S as follows:

$$f(x,y,z) \equiv \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \qquad ((x,y,z) \in S)$$

Show that there is a point (u,v,w) in S at which f achieves its minimum value. Find such a point and compute the minimum value of f.

08° Let a, b, and c be real numbers for which $a^2 + b^2 + c^2 = 1$. Let A be the antisymmetric matrix defined as follows

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Recall that:

$$exp(tA) = I + sin(t)A + (1 - cos(t))A^2$$

Verify that:

$$\frac{d}{dt}exp(tA) = Aexp(tA)$$