CURVATURE LITE

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1 Surfaces

01° Let U be a region in $\mathbf{R}^2$ and let H be an injective mapping carrying U to $\mathbf{R}^3$. Let $S := H(U)$ be the range of H, a subset of $\mathbf{R}^3$. We will refer to S as a *surface* in $\mathbf{R}^3$, *parametrized* by H. We will represent members of $\mathbf{R}^2$ as follows:

$$u = (u^1, u^2)$$

and members of $\mathbf{R}^3$ as follows:

$$x = (x^1, x^2, x^3)$$

Now the mapping H can be expressed in the following form:

$$(1) \qquad (u^1, u^2) = u \longrightarrow H(u) = x = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$$

We will represent the total derivative of H at u as follows:

$$DH(u) = \begin{pmatrix} H_1^1(u) & H_2^1(u) \\ H_1^2(u) & H_2^2(u) \\ H_1^3(u) & H_2^3(u) \end{pmatrix}$$

which is to say that:

$$(2) \qquad H_j^a(u^1, u^2) := \frac{\partial x^a}{\partial u^j}(u^1, u^2) \qquad (1 \leq j \leq 2,\ 1 \leq a \leq 3)$$

We require that, for each u in U, the column vectors:

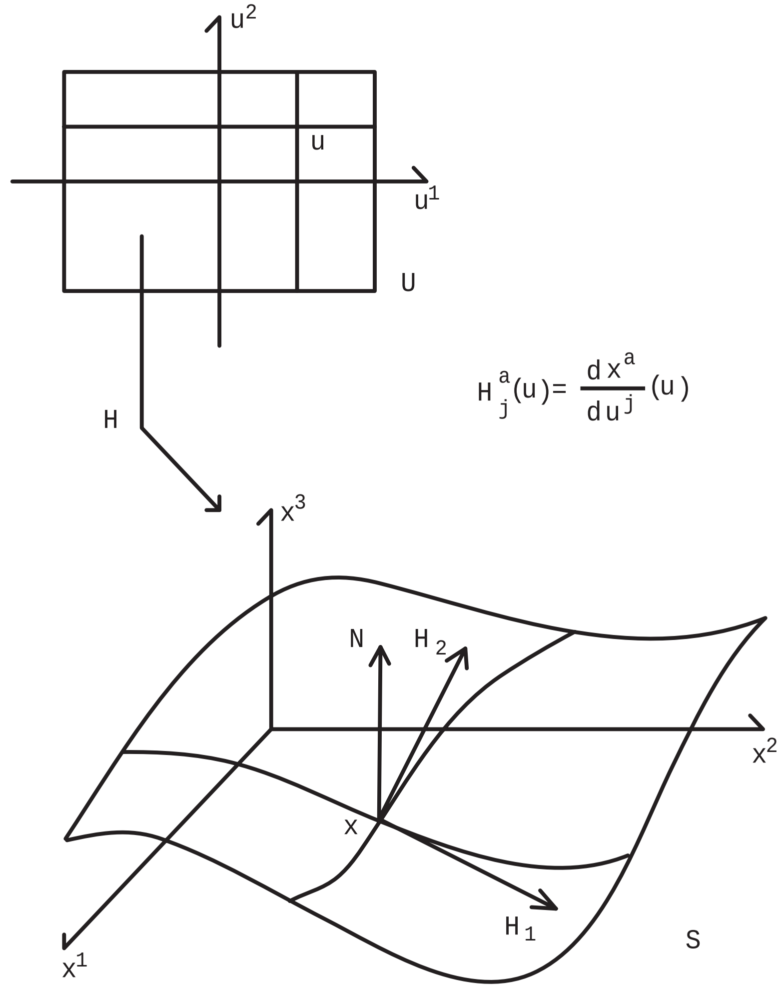
$$H_1(u) := \begin{pmatrix} H_1^1(u) \\ H_1^2(u) \\ H_1^3(u) \end{pmatrix} \quad \text{and} \quad H_2(u) := \begin{pmatrix} H_2^1(u) \\ H_2^2(u) \\ H_2^3(u) \end{pmatrix}$$

be linearly independent, which is to say that:

$$H_1(u) \times H_2(u) \neq 0$$

02° Let $N(u)$ be the unit vector normal to the surface S at the point $H(u)$:

(3)
$$N(u) := \frac{1}{\|H_1(u) \times H_2(u)\|}.(H_1(u) \times H_2(u))$$



03° We define the *first fundamental form* G for the surface S as follows:

$$G(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix}$$

where:

$$G_{k\ell}(u) := H_k(u) \bullet H_\ell(u) \qquad (1 \leq k \leq 2,\ 1 \leq \ell \leq 2) \tag{4}$$

One should note that $G(u)$ is a symmetric positive definite matrix.

04° We plan to describe the various metric properties of the surface S, such as the length of a curve in S, the area of a subset of S, and the curvature of S at a point. We will show that these properties can all be expressed in terms of the first fundamental form. This fact releases us from the view that, in general, a surface must lie in $\mathbf{R}^3$. We may focus our attention upon the region U in $\mathbf{R}^2$ and the first fundamental form G:

$$G(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix}$$

with which it has in some fashion been supplied. We may then proceed to calculate the various metric properties of U in terms of G.

05° Now let J be an open interval in $\mathbf{R}$ and let Γ be a mapping carrying J to $\mathbf{R}^3$ such that the range $C := \Gamma(J)$ of Γ is a subset of the surface S. We require that, for each t in J, $D\Gamma(t) \neq 0$. We shall refer to C as a *curve* in S, *parametrized* by Γ. Of course, we may introduce the mapping γ carrying J to U:

$$t \longrightarrow \gamma(t) = u = (u^1(t), u^2(t))$$

such that:

$$\begin{aligned}(\Gamma^1(t), \Gamma^2(t), \Gamma^3(t)) &= \Gamma(t) \\ &= H(\gamma(t)) \\ &= (H^1(u^1(t), u^2(t)), H^2(u^1(t), u^2(t)), H^3(u^1(t), u^2(t)))\end{aligned}$$

The mapping γ describes the given curve C in terms of the parameters u^1 and u^2. By the Chain Rule, we have:

$$D\Gamma(t) = DH(\gamma(t))D\gamma(t)$$

Hence:

$$\frac{d\Gamma}{dt}(t) = \frac{du^j}{dt}(t).H_j(\gamma(t)) \tag{5}$$

For the latter relation, we have invoked the *summation convention*, which directs that indices which appear in a given expression both "up" and "down" shall be summation indices running through their given range (in this case, from 1 to 2). In turn:

$$\|\frac{d\Gamma}{dt}(t)\|^2 = \frac{du^k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du^\ell}{dt}(t)$$

Now we may proceed to calculate the *length* of the segment of the curve C in S from $\Gamma(t')$ to $\Gamma(t'')$:

$$\text{(6)} \qquad \int_{t'}^{t''} \|D\Gamma(t)\|dt = \int_{t'}^{t''} \sqrt{\frac{du^k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du^\ell}{dt}(t)}dt$$

where t' and t'' are any numbers in J for which $t' \leq t''$. We are led to interpret:

$$\text{(7)} \qquad \|V\| := \sqrt{V^k G_{k\ell}(u)V^\ell}$$

as the *length* of the tangent vector:

$$V := \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

to U at u, and to interpret:

$$\int_{t'}^{t''} \sqrt{\frac{du^k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du^\ell}{dt}(t)}dt$$

as the *length* of the segment of the curve γ in U from $\gamma(t')$ to $\gamma(t'')$. More generally, we interpret:

$$\text{(8)} \qquad V \circ W := V^k G_{k\ell}(u)W^\ell$$

as the *inner product* of the vectors:

$$V = \begin{pmatrix} V^1 \\ V^2 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}$$

in $\mathbf{R}^2$, tangent to U at u.

06° We may also proceed to calculate the *area* of a subset T of S, as follows. We first present T as $T = H(V)$, where V is a subset of U. We then equate the *area* of T with the following double integral:

$$\text{(9)} \qquad area(T) := \int\int_V \|H_1(u^1, u^2) \times H_2(u^1, u^2)\|du^1du^2$$

Since:

$$\|H_1(u) \times H_2(u)\|^2 = G_{11}(u)G_{22}(u) - G_{21}(u)G_{12}(u) =: g(u)$$

we interpret:

$$area(V) := \int\int_V \sqrt{g(u^1, u^2)}du^1 du^2 \tag{10}$$

as the area of the subset V of U.

2 Curvature

07° Let us consider a particular point $\bar{P}$:

$$\bar{P} = (\bar{x}^1, \bar{x}^2, \bar{x}^3) = H(\bar{u}^1, \bar{u}^2)$$

in the surface S. We plan to describe the *curvature* of S at $\bar{P}$. To that end, let us consider a curve C in S containing $\bar{P}$. The curvature of C at $\bar{P}$ derives in part from the bending of C within S and in part from the bending of S itself. One may refer to the former as the *internal* bending of C and to the latter as the *external* bending. One may say that the internal bending is a matter of free choice but that the external bending is forced upon the curve by the structure of the surface. Among all curves C in S containing $\bar{P}$, we may consider those for which the external bending is minimum and those for which it is maximum. By definition, the *gaussian curvature* of the surface S at the point $\bar{P}$ is the product of these two extreme values.

08° Let J be an open interval in $\mathbf{R}$ and let Γ be a mapping carrying J to $\mathbf{R}^3$ such that $C := \Gamma(J)$. As usual, we require that, for each t in J, $D\Gamma(t) \neq 0$. For convenience, let 0 be in J and let $\Gamma(0) = \bar{P}$. In turn, let γ be the mapping carrying J to U:

$$t \longrightarrow \gamma(t) = u = (u^1(t), u^2(t))$$

such that:

$$\begin{aligned}(\Gamma^1(t), \Gamma^2(t), \Gamma^3(t)) &= \Gamma(t)\\ &= H(\gamma(t))\\ &= (H^1(u^1(t), u^2(t)), H^2(u^1(t), u^2(t)), H^3(u^1(t), u^2(t)))\end{aligned}$$

Of course, $\gamma(0) = \bar{u} = (\bar{u}^1, \bar{u}^2)$. We have:

$$\frac{d\Gamma}{dt}(t) = \frac{du^j}{dt}(t).H_j(\gamma(t))$$

and:
$$\frac{d^2\Gamma}{dt^2}(t) = \frac{d^2u^j}{dt^2}(t).H_j(\gamma(t)) + \frac{du^k}{dt}(t)\frac{du^\ell}{dt}(t).H_{k\ell}(\gamma(t))$$

where:

$$H_{k\ell}(u) := \frac{\partial^2 H}{\partial u^k \partial u^\ell}(u) \tag{11}$$

Now we may introduce functions $K^j_{k\ell}$ and $L_{k\ell}$ such that:

$$H_{k\ell}(u) = K^j_{k\ell}(u).H_j(u) + L_{k\ell}(u).N(u) \tag{12}$$

The foregoing relations are called *Gauss' Equations*. One should note carefully that:

$$L_{k\ell}(u) = H_{k\ell}(u) \bullet N(u) \tag{13}$$

One refers to L:
$$L(u) = \begin{pmatrix} L_{11}(u) & L_{12}(u) \\ L_{21}(u) & L_{22}(u) \end{pmatrix}$$

as the *second fundamental form* for the surface S. One refers to K^1 and K^2:

$$K^1(u) = \begin{pmatrix} K^1_{11}(u) & K^1_{12}(u) \\ K^1_{21}(u) & K^1_{22}(u) \end{pmatrix} \quad \text{and} \quad K^2(u) = \begin{pmatrix} K^2_{11}(u) & K^2_{12}(u) \\ K^2_{21}(u) & K^2_{22}(u) \end{pmatrix}$$

as the *connection coefficients* for S. Finally, we obtain:

$$\frac{d^2\Gamma}{dt^2}(t) = A^j(t).H_j(\gamma(t)) + B(t).N(\gamma(t)) \tag{14}$$

where:

$$A^j(t) := \frac{d^2u^j}{dt^2}(t) + \frac{du^k}{dt}K^j_{k\ell}(\gamma(t))(t)\frac{du^\ell}{dt}(t) \tag{15}$$

and:

$$B(t) := \frac{du^k}{dt}(t)L_{k\ell}(\gamma(t))\frac{du^\ell}{dt}(t) \tag{16}$$

Clearly:
$$A^j(t).H_j(\gamma(t))$$

is tangent to S at $H(u)$. It represents the internal bending of C at $H(u)$. Moreover:
$$B(t).N(\gamma(t))$$

is normal to S at $H(u)$. It represents the external bending of C at $H(u)$.

09° At this point, we are interested in the value of $B(0)$:

$$B(0) = \frac{du^k}{dt}(0) L_{k\ell}(\bar{u}) \frac{du^\ell}{dt}(0) \tag{17}$$

since it measures the "external bending" of C at $\bar{P}$. To set the scale of computation, we require that C be parametrized by arc length. The effect of this requirement is to force:

$$\frac{du^k}{dt}(t) G_{k\ell}(\gamma(t)) \frac{du^\ell}{dt}(t) = 1$$

In particular:

$$\frac{du^k}{dt}(0) G_{k\ell}(\bar{u}) \frac{du^\ell}{dt}(0) = 1 \tag{18}$$

Now we wish to study the minimum and maximum values of the quantity:

$$V^k L_{k\ell}(\bar{u}) V^\ell$$

where V is any vector in $\mathbf{R}^2$ meeting the condition:

$$V^k G_{k\ell}(\bar{u}) V^\ell = 1$$

The product of these extreme values is the gaussian curvature for S at $\bar{P}$.

10° Here is our problem. We have two symmetric matrices:

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

and:

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

The latter is positive definite. These matrices define functions ("quadratic forms") as follows:

$$\lambda(V) := V^k L_{k\ell} V^\ell = \begin{pmatrix} V^1 & V^2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

and:

$$\gamma(V) := V^k G_{k\ell} V^\ell = \begin{pmatrix} V^1 & V^2 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

We wish to calculate the product of the minimum and the maximum values of the quantity $\lambda(V)$, subject to the condition $\gamma(V) = 1$. By "diagonalizing" the quadratic form L relative to the (positive definite) quadratic form G, one can show that the foregoing product equals:

$$\frac{L_{11}L_{22} - L_{21}L_{12}}{G_{11}G_{22} - G_{21}G_{12}}$$

Accordingly, we define the curvature of the surface S at the point $\bar{P}$ to be:

$$\begin{aligned} \kappa_S(\bar{P}) :&= \frac{L_{11}(\bar{u})L_{22}(\bar{u}) - L_{21}(\bar{u})L_{12}(\bar{u})}{G_{11}(\bar{u})G_{22}(\bar{u}) - G_{21}(\bar{u})G_{12}(\bar{u})} \\ &= \frac{L_{11}(\bar{u})L_{22}(\bar{u}) - L_{21}(\bar{u})L_{12}(\bar{u})}{g(\bar{u})} \end{aligned} \tag{19}$$